A KNOWLEDGE STRUCTURE FOR THE ARITHMETIC MEAN: RELATIONSHIPS BETWEEN STATISTICAL CONCEPTUALIZATIONS AND MATHEMATICAL CONCEPTS

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This study examined cognitive relationships between the fair-share and center-of-balance conceptualizations of the arithmetic mean. It also hypothesized the use of these conceptualizations as blending spaces for the mathematical and statistical domains within a proposed knowledge structure for the arithmetic mean.

Twenty-nine undergraduate liberal arts students completed pre/post verbal protocols with written solutions to arithmetic mean problems. The problems emphasized either the fair-share or center-of-balance conceptualization, or mathematical concepts related to the arithmetic mean. The participants were divided into three groups: those that received fair-share instruction, those that received center-of-balance instruction, and a control group.

The data was analyzed using statistical methods, including contingency tables and ANCOVA, to investigate the effects fair-share and center-of-balance instruction had on knowledge of fair-share, center-of-balance, and mathematical concepts regarding the arithmetic mean. A qualitative analysis of the verbal protocols helped explain any statistically significant connection between the fair-share and center-of-balance conceptualizations, or between either conceptualization and mathematical concepts related to the arithmetic mean.

Analysis of the data indicated participants increased their knowledge of the fair-share conceptualization after receiving instruction that was focused on center-of-balance. Similarly,

participants increased their knowledge of the center-of-balance conceptualization after receiving instruction that was focused on fair-share. In either case, the concept, ‘the sum of the deviations from the mean is zero,’ was used to transfer knowledge between the conceptualizations.

In addition, instruction in either the fair-share or center-of-balance conceptualization increased knowledge of the mathematical concepts related to the arithmetic mean. However, only specific mathematical concepts were impacted by each of the conceptualizations.

The results suggest that both the fair-share and center-of-balance conceptualizations are pertinent to pedagogical decisions regarding the arithmetic mean. Furthermore, the concept, ‘the sum of the deviations from the mean is zero,’ is a viable cognitive connection between the fair-share and center-of-balance conceptualizations.

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To those who sacrificed so that I could succeed, especially,

Jennifer, Arianne, Brielle, Mom, Dad, and Louise

1.0 INTRODUCTION

Research in mathematics education has long provided valuable descriptions of knowledge structures detailing mathematical understanding. Studies have examined, for example, the relationships between different types of mathematical knowledge (e.g. Rittle-Johnson, Siegler, & Alibali, 2001; Shulman, 1986), the relationships between representations of knowledge (e.g. Bruner, 1964; Lesh, Post & Behr, 1987), the relationships amongst content within a mathematical field (e.g. McDonald, 1989), and the relationships between different constructs of a mathematical topic (e.g. Kieren, 1988; Williams, 1998). Statistical concepts, such as the arithmetic mean, are not just subject to the four relations above; they also encompass relationships between mathematical content and statistical concepts. Research to investigate these unique relationships is increasing because of a deliberate effort in a relatively new field of study, statistics education. Because of the symbiotic relationship between the fields of mathematics and statistics, it is reasonable to draw on the significant insights and findings from the discipline of mathematics education and apply them, or at least use them as a basis, for research and advancement in statistics education. A vital need in the field of statistics education is the refinement of existing knowledge structures that acknowledge the conceptual differences, yet preserve the inherent relationships between mathematics and statistics (Groth, 2007). Developing concept-specific knowledge structures that refine the existing discipline-level frameworks can help deepen the understanding of the conceptual relationships that constitute knowledge of statistics.

One area of statistical study in which better understanding of the learners' knowledge could lead to innovations in the pedagogical process is the arithmetic mean. Its seemingly simple tie to a standard mathematical procedure masks its statistical profundity and, ultimately, the misconceptions and lack of conceptual understanding many students encounter. Its statistical role makes it widely used (or misused) in most academic and professional disciplines, as well as in everyday life.

Previously researched knowledge models (Groth, 2007; Jones et al., 2000; Mooney, 2002) along with past studies (e.g. Cai, 1998; Cobb & Moore, 1997, Mokros & Russell, 1995; Strauss & Bichler, 1988) have identified both mathematical and statistical elements of the arithmetic mean. Two concepts related to the arithmetic mean, *center-of-balance* and *fair-share*, are connected to both its place in mathematics and its place in statistics. The center-of-balance conceptualization views the arithmetic mean as the point of balance of the data (e.g. Hardiman, Well, & Pollatsek, 1984); while the fair-share model views the arithmetic mean as an equal distribution of the data (e.g. Cai & Moyer, 1995; George, 1995; Mokros & Russell, 1995). While past studies have identified models depicting fair-share and center-of-balance and their use in improving classroom instruction; they have not reported on the cognitive relationships, if they exist, between fair-share and center-of-balance, or the cognitive relationships of the mathematical and statistical elements of the arithmetic mean. These relationships are the focus of this study.

1.1 BACKGROUND

1.1.1 Differences between Mathematics and Statistics

Statisticians and statistics educators have long argued that statistics is not a branch of mathematics, but rather a branch of science that utilizes mathematical tools, similar to economics and physics, to explore its own concepts (Cobb & Moore, 1997; Hand, 1998). Mathematical knowledge is not the only knowledge necessary to understand statistical concepts. Statistics is a discipline that involves applying statistical concepts and techniques, often distinct from those in mathematics, to other fields of study in order to solve real world problems. One major difference between statistics and mathematics is the role of context. In mathematics, problem context usually needs to be "boiled off" to get at the root of the abstract mathematical structure. On the other hand, in statistics, the problem context along with the components of the application domain provides meaning to the data analysis (Cobb & Moore, 1997; Hand, 1998). In statistics, the "data are not just numbers; they are numbers with a context" (Cobb & Moore, 1997, p. 801). A second dissimilarity between mathematics and statistics is their respective academic spaces. Statistics is a methodological discipline rather than a core substantive area like mathematics (Cobb & Moore, 2000). It is an interdisciplinary science with links to many different fields of study and application. A third distinction concerns the sense of variability, inference, and interpretation which is essential to statistical analysis but absent from mathematical principles. These profound differences in the disciplines suggest the knowledge needed to understand statistics includes both mathematical knowledge and knowledge proprietary to statistics.

Hand (1998) raises the question as to why statistics is taught; is it to develop students who can advance statistical methodology or is it to develop students who can carry out effective statistical data analyses? Because the vast majority of students learning statistics fall into the latter group, a joint committee of the American Statistical Association (ASA) and the Mathematical Association of America (MAA) contend instruction of statistics should emphasize statistical ideas and concepts, including the importance of data, the omnipresence of variability, and both the quantification and explanation of data and variability (Cobb & Moore, 1997). Some confusion as to the place of statistics education arises from the fact that in K-12 education, and in many small colleges and universities, statistics is taught by mathematics teachers and mathematicians; many of whom value it as a mathematical discipline.

1.1.2 Relationship between Mathematical and Statistical Knowledge

Previous research in mathematics education (see e.g. Resnick, 1983; Cobb et al., 1991) has advocated the need for detailed cognitive models of students' reasoning that help guide mathematical pedagogy. "According to Cobb and Resnick, such cognitive models should incorporate key elements of a content domain and the process by which students grow in their understanding of the content within that domain" (Jones, Langrall, Mooney, & Thornton, 2004, p. 101). The call for research-based cognitive models has been answered by cognitive psychologists, mathematics educators, as well as statistics educators.

The authors Jones et al. (2000) and Mooney (2002) indicate one of four major components in statistical understanding and reasoning is data analysis. They also indicate an important attribute of comprehending data analysis is the cognitive relationship between statistical knowledge and mathematical knowledge. The related research by Jones et al. and Mooney consisted of qualitative analyses of interviews based on a statistical thinking protocol. The protocol incorporated data exploration tasks, open-ended questions, and subsequent probes to these questions. The results of their studies indicated that understanding of statistics occurs as two sequential "cycles" of statistical reasoning. The first cycle "deals with the conceptual development of statistical concepts, while the second cycle…deals with the application of statistical and mathematical concepts and procedures that have already been learned" (Jones et al., 2004, p. 108). The two-cycles of development indicate the importance of cultivating a new statistical concept prior to advancing its mathematical basis or procedural application. The significance of this result as it pertains to the specific statistical concept of the arithmetic mean will be noted later in chapter two.

Groth (2007), in a research commentary on statistical knowledge needed for teaching, argues that it is imperative that the differences between the knowledge structures necessary for understanding statistics and the knowledge structures necessary for understanding mathematics be explicitly differentiated in order to further the research on statistical knowledge structures. He further contends that, although mathematical and statistical knowledge are different, engaging in many meaningful statistical activities involves the simultaneous activation of both mathematical and statistical knowledge.

Examining previous research can both distinguish mathematical and statistical knowledge and provide evidence of their symbiotic relationship in statistics. Descriptive statistics, such as the arithmetic mean, are fundamental concepts in statistical data analysis. It is therefore reasonable to suspect components of data analysis, such as the arithmetic mean, also possess both mathematical and statistical attributes and the subsequent integrating cognitive links.

1.1.3 Arithmetic Mean as a Subject of Study

The *Curriculum and Evaluation Standards* (1989) and *Principal and Standards for School Mathematics* (2000) documents from the National Council of Teacher of Mathematics (NCTM) suggest that students should have a "solid understanding" of mean as a measure of center. The significant role of the arithmetic mean among averages is also supported by the American Mathematical Association of Two-Year Colleges' (AMATYC) standards document, *Crossroads* (1995), the ASA endorsed, *Guidelines for Assessment and Instruction in Statistics Education (GAISE)* report (Franklin et al., 2007), as well as in other policy-making documents (e.g. *Adding It Up: Helping Children Learn Mathematics* (Kilpatrick, Swafford & Findell, 2001), *Victorian Essential Learning Standards: Discipline Based Learning Strand Mathematics* (VCAA, 2005)). These documents emphasize the importance of understanding measures of central tendency, including the arithmetic mean as the most commonly occurring measure, and its necessity in shaping a statistically literate society.

Different from many of the other descriptions of average, the arithmetic mean has uses in statistics beyond the suggestion of central tendency. It is utilized, for example, in calculating other statistics such as the standard deviation, creating formulas for distributions such as the Poisson and normal, finding confidence intervals, and testing hypotheses.

The arithmetic mean can also inform or model concepts outside of statistics. In a physical sense, the arithmetic mean can be thought of as a center of gravity. From the mean of a data set we can think of the average distance the data points are from the mean as standard deviation. The square of standard deviation (i.e. variance) is analogous to the moment of inertia in the physical model. The formulas for calculation correspond exactly (N. Pfenning, personal communication, May 10, 2008). The ability to mathematically model concepts in physics to concepts in statistics signifies the potential significance of comprehensively understanding all aspects of the arithmetic mean.

The cross-disciplinary nature of the arithmetic mean makes it conceptually constructive in many disciplines of study, including statistics, mathematics, and physics, and its use as a statistical tool makes it omnipresent in educational, vocational, and recreational settings. The arithmetic mean's diversity has fostered research aimed at finding an understanding of how students arrive at their knowledge base for the arithmetic mean and the instructional techniques that promote its conceptual learning.

The arithmetic mean is one of many different kinds of averages used to describe the center or representative value of a data set. This seemingly simple calculation is actually a relatively complex concept that is most often developed as an "add-them-up-and-then-divide" mathematical procedure, rather than as a statistically representative concept. Developing the concept as a statistical representation of a data set is often encumbered by early exposure to the rote algorithmic procedure used to calculate the mean (Mokros & Russell, 1995; Konold & Higgins, 2003). Unlike many mathematical concepts, the conceptual and procedural knowledge of the arithmetic mean do not seem to develop in an iterative or "hand-over-hand" manner (see Hiebert & Lefevre, 1986, for discussion on conceptual and procedural knowledge; and Rittle-Johnson & Alibali, 1999, for discussion on iterative growth of knowledge). A plausible explanation of this inconsistency is much of the conceptual knowledge related to the arithmetic mean is not purely mathematical; rather it is a combination of conceptual knowledge of mathematics and conceptual knowledge of statistics.

1.2 PURPOSE OF STUDY

This study has a dual purpose in examining knowledge as it relates to the arithmetic mean. First, it uses the existing literature as a basis to propose knowledge relationships specific to the arithmetic mean. In doing so, the study addresses the concerns of Resnick (1983), Cobb et al. (1991), and Groth (2007) to develop cognitive models of reasoning that include research on the growth and development of elements within knowledge domains. The study delineates the domains and identifies the concepts of fair-share and center-of-balance as cognitive blending spaces between the mathematical and statistical domains. Second, the study aims to examine several of the previously unsubstantiated cognitive relationships that exist within the hypothesized knowledge structure. In particular, this study seeks to address the cognitive link between the notions of fair-share and center-of-balance, and the cognitive link between these notions and the mathematical domain of the arithmetic mean.

1.2.1 Research Questions

The following research questions are designed to further the understanding of knowledge related to the arithmetic mean by investigating the cognitive role of the fair-share and center-of-balance conceptualizations.

1) How is knowledge of fair-share and center-of-balance cognitively related to one another? In particular,

    a) What effect does instruction of the fair-share conceptualization of the arithmetic mean have on knowledge of the center-of-balance conceptualization?

    b) What effect does instruction of the center-of-balance conceptualization of the arithmetic mean have on knowledge of the fair-share conceptualization?

2) How is knowledge of fair-share and center-of-balance cognitively related to the mathematical domain? In particular,

   a) What effect does instruction of the fair-share conceptualization of the arithmetic mean have on knowledge of mathematical concepts associated with the arithmetic mean?

   b) What effect does instruction of the center-of-balance conceptualization of the arithmetic mean have on knowledge of mathematical concepts associated with the arithmetic mean?

The first research question is designed to investigate any potential connection between the notions of fair-share and center-of-balance as they relate to the arithmetic mean. The second research question focuses on the impact instruction of the fair-share or center-of-balance conceptualizations has on the mathematical concepts associated with the arithmetic mean.

1.3 CONTRIBUTION TO THE DISCIPLINES

This study contributes to the fields of mathematics education and statistics education in a number of ways. First, it builds on and refines existing research aimed at modeling the development of statistical reasoning. This current study refines discipline-level statistical knowledge models by focusing on the cognitive relationships of a specific statistical concept, the arithmetic mean. Understanding the dynamics of the cognitive structure of a particular statistical concept may better inform instructional decisions with regards to teaching the arithmetic mean.

Second, the study examines the relationship between two representative conceptualizations that illustrate the arithmetic mean, fair-share and center-of-balance. While particulars concerning each of these representations have been previously studied, little research has examined any possible relationship or link between them (i.e. how they inform each other). Unfortunately, in many academic settings, particularly in colleges and universities, only a limited amount of time is available to develop a mathematical or statistical concept like the arithmetic mean. A particular conceptualization may inform knowledge of the other and provide a more robust and/or efficient way of conceptualizing the abstract nature of the arithmetic mean.

Third, the study provides insight into the connection between knowledge of mathematics and knowledge of statistics through the concept of the arithmetic mean. Statistics provides mathematics with a basis of contextually rich real-world problems that can be used to contextualize the mathematics. Conversely, mathematics is a tool utilized by statistics to quantify statistical concepts. An understanding of their inter-disciplinary knowledge connections may advance the pedagogical symbiosis between mathematic and statistics.

The arithmetic mean has a place in the often amalgamated fields of statistics education and mathematics education. Understanding how knowledge is developed within each discipline and how knowledge is connected across the disciplines influences pedagogical decisions in both branches of learning. By researching the knowledge relationships of the arithmetic mean, an interdisciplinary concept, this study adds to the literature in the fields of both mathematics education and statistics education

1.4 LIMITATIONS OF STUDY

Several limitations should be considered as they pertain to this study. First, the sample for the study was chosen because the academic level of the participants was of particular interest to the researcher and it represented a wide range of mathematical knowledge. The sample of students enrolled in a liberal arts mathematics course at a small private university may not generalize to students of different ages or different academic interests. Second, the verbal protocols, as most data gathering instruments, only reveal the knowledge participants select to apply to a particular task, not necessarily the total knowledge they possess for that concept. The verbal protocols provide resonant detail regarding the knowledge used by the participants. The small sample size of the protocols may not produce the typically accepted power for statistical analysis, but the sample size was chosen to balance the quantitative and qualitative aspects of the study. In particular, the richness of the data provided by the verbal protocols should compensate for the statistical deficiency of the sample size.

1.5 ORGANIZATION OF THE DISSERTATION

This document is organized into five chapters. Chapter one provides a motivation for the research, including a discussion on the relationships between mathematics and statistics, and the importance of the arithmetic mean as a subject of study. The chapter then offers an overview of this research study, including its purpose, contribution to mathematics education and statistics education, limitations, and organization. Chapter two reviews the relevant literature in order to accomplish the following objectives: report on the development of knowledge regarding the arithmetic mean, develop the knowledge elements and domains associated with the arithmetic mean, connect the notions of fair-share and center-of-balance to the mathematical and statistical domains, and communicate the pedagogical issues surrounding instruction of the arithmetic mean. Chapter three considers the literature germane to the methodology and situates this study in those principles. This includes an account of the demographics, instrument, and procedure along with details of data collection, data coding, and data analysis. The theoretical discussion is accompanied by the practical results of pilot work. Chapter four presents the results of the analyzed data. Chapter five offers a discussion of the research findings, recommendations based on the research findings, and areas of future study revealed by the literature review and research study.

2.0 REVIEW OF LITERATURE

This review synthesizes the relevant literature to situate the concepts of *fair-share* and *center-of-balance* in a knowledge structure for the arithmetic mean and investigates the relationships between these concepts and domains within that structure. First, the development of knowledge for notions of average and the arithmetic mean is examined. This section includes situating the arithmetic mean among the realm of averages and delineating its development of understanding. Second, the relatively complex dual nature of the arithmetic mean is presented and its mathematical and statistical components are expounded. Third, a discussion defining the concepts of fair-share and center-of-balance and relating each concept to both mathematical and statistical ideas is presented. Fourth, a knowledge structure for the arithmetic mean is hypothesized utilizing the existing literature base. This includes establishing two domains of knowledge, mathematical and statistical, that house the concepts applicable to the arithmetic mean, and elaborating on how knowledge between the domains is related by the fair-share and center-of-balance conceptualizations. Finally, a review of the literature examining instructional studies associated with the arithmetic mean is included to explore the implications that the knowledge structure for the arithmetic mean has on pedagogical issues.

2.1 DEVELOPING UNDERSTANDING OF THE ARITHMETIC MEAN

The words average or central tendency are used in statistics to describe a single notion or representative value that describes the center, middle, or expected value of a larger set. There are many different kinds of averages that can be properly chosen based on informed analysis of the data.[1]

2.1.1 Averages and Their Development

The scholastic development of average most often begins in the primary grades with the concept of mode, followed by midrange and median. Studies indicate that when early primary grade students are first introduced to data sets they have difficulties seeing the data as a whole and focus on the aspects of the individual data points (Hancock, Kaput & Goldsmith, 1992). Similarly, Lehrer and Schauble (2000) found that students in first and second grade were largely unable to use classification techniques to represent groups of drawings, but students in fourth and fifth grade were able to appreciate the value of assigning dimensional attributes or representations for categorizing the drawings. These studies indicate the idea of recognizing trends or representativeness of data occurs for most students around the third grade.

It is through early life experiences that children begin to build an intuitive view of average based on qualitative notions of typicality or representativeness. In third grade most

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[1] While mean, median, mode, and midrange are the most common measures of central tendency, other more specialized measures exist, such as:

Harmonic mean--used for finding "average per"
Geometric mean--used for finding averages of percentage, ratios, indexes, and growth rates
Quadratic mean--use in physical sciences and electronics.

Many other specialized means exist to measure specific discrete data as well as measures in calculus to measure continuous data and functions students have encountered the term "average" from experiences with grade averages or average temperatures (Konold & Higgins, 2003). This intuitive sense of average is most often expressed as "most" or "middle," which are ideas related to the formal averaging concepts of mode and median, respectively (Watson & Moritz, 2000). At this stage of development many students blend notions of average to form *ideal averages*. According to Konold & Higgins (2003), ideal averages have four properties: (a) an actual value in the data set, (b) the most frequently occurring value in the data set, (c) located midway between two extreme values in the data set, and (d) relatively close to all other values in the data set. Two examples of ideal averages are the *middle-clump*, a cluster of values in the heart of a distribution, and the *modal-clump*, a central range of values that not only indicates central tendency, but also some sense of the data's distribution (Konold et al., 2002; Russell, Schifter & Bastable, 2002).

Interestingly, the most commonly used descriptive statistic, the arithmetic mean, is absent from the sense of average described by primary grade students. Students do not make use of the arithmetic mean, or an intrinsic meaning of it such as fair-share or center-of-balance, until it is formally introduced in the fourth or fifth grade in the United States (in the sixth grade in Australia) and revisited each year until the eighth grade with increasing procedural complexity (Watson & Moritz, 2000). The arithmetic mean may also appear in high school curricula, such as in a general math and algebra course. The notion of average may also be introduced as centroids in a geometry course, or as means of continuous probability functions in Calculus. As little as ten years ago a student's first encounter with a traditional statistics course was in college; now some high schools offer statistics in their mathematics curriculum as an elective, Advanced Placement, or College in High School course.

While several notions of average are conceptually developed by students themselves at a relatively early age, the arithmetic mean is less transparent and the conceptual underpinnings necessary to sensibly use it are surprisingly difficult (Konold & Higgins, 2003). Students intuitively construct a sense of mode and median before being formally introduced to the concepts and procedures for finding them. Conversely, students rarely have the opportunity or insight to do the same with regards to the arithmetic mean.

2.1.2 Arithmetic Mean and Its Development

Since the arithmetic mean is an entity in statistics, it is reasonable to parallel the role of mathematics in statistics to the role of mathematics in the arithmetic mean. That is, the arithmetic mean is a statistical concept defined outside of the field of mathematics, but which uses mathematics extensively in its calculation (see section 1.1). The statistical and mathematical attributes of the arithmetic mean can be uniquely defined and then integrated to understand and thoughtfully apply the arithmetic mean. The arithmetic mean is “a mathematical construction that represents certain relationships in the data” (Russell & Mokros, 1996, p. 362). It is a mathematical abstraction that denotes the statistical representativeness of the data.

Given this complexity, it is not surprising that research concerning knowledge of the arithmetic mean has indicated that while most students beyond the fourth grade are capable of applying an “add-and-then-divide” procedure to find a mean, many have difficulty conceptually understanding what it represents (Russell & Mokros, 1996). This difficulty in conceptually understanding the arithmetic mean was also found to be prevalent in middle grade students (Mokros & Russell, 1995; Cai, 1998; McGatha, Cobb, & McClain, 2002), and among college students (Pollatsek, Lima & Well, 1981; Groth & Bergner, 2006).

A study that investigated students' understanding of the arithmetic mean for three education levels, primary, middle, and college, found improved performance with age on problems associated with the four component properties of the mean, but a general lack of understanding of the abstract and representative aspects of the mean (Leon & Zawojewski, 1990). The four component properties were (a) the mean is located between extreme values of a data set, (b) the sum of the deviations about the mean is zero, (c) a value of zero in the data set must be accounted for, and (d) the average value is representative of the values that were averaged. Watson and Moritz (2000) conducted a longitudinal study of students in third through eleventh grades in order to examine the development of the idea of average. One-hundred-and-thirty-seven interviews were conducted to collect problem solving data. An analysis of the data indicated the ability to correctly apply the arithmetic mean formula increased with grade, but the language students used to describe average showed little growth for the notion of representativeness. Mokros and Russell (1995) used extensive task-based interviews of twenty-one students in fourth through eighth grade. They found most students knew the add-and-then-divide algorithm but related it to limited context and were unable to use it in any meaningful way. They further concluded that "children construct the idea of representativeness through many encounters with a variety of real data sets" (p. 37).

While overall conceptual knowledge of the arithmetic mean (i.e. *representing* the data) is lacking, the research indicates that procedural knowledge of the arithmetic mean (i.e. *computing* the mean) is retained from the time it is introduced in the fourth grade until at least a student's college years. These studies indicate as age increases so does the ability to apply the arithmetic mean formula. Unfortunately, any increased conceptual understanding of the arithmetic mean during a student's education, unless the student is exposed to a curriculum or teacher that conceptually develops the mean (see section 2.5), most likely corresponds to exposure to contextual situations outside of the classroom.

The traditional instructional technique[2] for teaching the arithmetic mean includes a simple introduction to the add-and-then-divide procedure inherent in the computational formula, $\bar{x} = \frac{\sum x_i}{n}$, and a succinct synopsis on measures of central tendency (Shaughnessy, 1992). From the time arithmetic mean is introduced, students tend to use this procedure most often to find an average of a data set regardless of their conceptual understanding of average (Groth & Bergner, 2006). There is now concern that this type of rote algorithmic instruction and application early in a student's development may cause a short-circuit in the student's reasoning. Mokros and Russell (1995) note, for example, that many students who had "sound informal ideas about average as a representative measure" may be developmentally impeded by the rote nature of the algorithm. The researchers suggest students should be "pulled away" from the rote algorithm in order to further a learner's understanding of average (p. 37). Many researchers who have investigated student use and understanding of mean suggest that less emphasis should be placed on teaching procedures in the primary grades and more emphasis should be placed on developing the conceptual understanding of representativeness (Konold & Higgins, 2003; Mokros & Russell, 1995). The notion of emphasizing the statistical conception of the arithmetic mean (i.e. representativeness) before applying the mathematical construct is consistent with the findings of Jones et al. (2000) and Mooney (2002) (see section 1.1.2). Their findings indicated statistical reasoning is developed in two cycles. The first cycle is an understanding of the statistical

[2] Reform curricula such as *Connected Mathematics* and *Investigations in Number, Data, and Space* apply a more conceptual approach to developing the arithmetic mean.

concept; the second cycle is the application of statistical and mathematical concepts and procedures that have already been learned.

The research indicates that an understanding of the arithmetic mean is best developed by addressing the statistical ideas associated with the arithmetic mean before presenting the mathematical procedures for calculating it. It is also clear that both the statistical and mathematical concepts pertaining to the arithmetic mean are vital in understanding and utilizing it. Because of the symbiotic relationship between statistics and mathematics, it is reasonable to draw upon the extensive research in mathematics education devoted to the development of procedural and conceptual knowledge to help explain the conceptual relationships of the mathematical knowledge associated with the arithmetic mean (see e.g. Hiebert & Lefevre, 1986). The following sections uniquely define the mathematical and statistical domains of knowledge associated with arithmetic mean.

2.2 MATHEMATICAL AND STATISTICAL KNOWLEDGE DOMAINS

Pollatsek, Lima, and Well's (1981) interviews with seventeen undergraduate students found an adequate schema for the arithmetic mean consists of three distinguishable types of knowledge: (a) *computational* knowledge which relates to the mathematical procedures and concepts necessary to compute the arithmetic mean, (b) *functional* knowledge which refers to the arithmetic mean as a real-world concept that is representative of a data set, and (c) *analog* knowledge which uses analogous concepts, in this case the arithmetic mean as a balancing point, to translate between equations and verbal descriptions. They combined think-aloud problem solving with follow-up interviews to gather data on arithmetic mean problems. A qualitative analysis of their data indicated that a lack of the arithmetic connections necessary to conceptually understand the mean leads to an inability to solve problems not posed as rote computations.

2.2.1 Mathematical Knowledge of the Arithmetic Mean

The mathematical knowledge necessary to understand, calculate and utilize the arithmetic mean is a subset of a student's complete mathematical knowledge and understanding. Pollatsek, Lima, and Well's (1981) coined this type of knowledge related to the arithmetic mean as *computational*. Each of the new procedures and concepts related to the arithmetic mean needs to be connected to previous knowledge and properly assimilated or accommodated within the existing web of knowledge. This new knowledge structure will allow a student to access the necessary mathematical procedures and concepts associated with the arithmetic mean.

Two areas of mathematics are relevant to the calculations and application of the arithmetic mean. *Arithmetic* is intrinsic to the mean formula and its calculation. *Algebra* is a necessary component for manipulating the arithmetic mean formula as well as for solving missing value problems. A better understanding of algebraic properties could help alleviate many of the misconceptions about the properties of groups as they apply to the arithmetic mean. The knowledge within each of these areas of mathematics is constructed by connecting a network of procedures and concepts. A comprehensive mathematical understanding of the arithmetic mean includes the interrelationships between arithmetic and algebra.

Knowledge specific to the arithmetic mean can be procedural, such as computing the mean using a formula, defining the relevant variables, or knowing the fact that the 'sum of the deviations from the mean is zero.' This knowledge may also be conceptual, such as mathematically understanding why the mean formula forces the sum of the deviations from the mean to be zero,[3] or using the properties of algebra to realize the mean is not a binary operation and therefore does not have an identity element; so the mean is influenced by numbers other than the average.

2.2.1.1 Mathematical Procedural Knowledge of Arithmetic Mean

The two-part definition of procedural knowledge, recognizing the correct use of syntax and symbols along with applying rules and algorithms (Hiebert & Leferve, 1986) is easily applicable to the arithmetic mean. The symbolic representation of arithmetic mean is universally accepted as $\bar{x}$ or μ. Further symbolic knowledge includes recognition of the variable *x* as representing values of data points and *n* as the number of data points. The second aspect of procedural knowledge involves the rules, algorithm, and procedures of calculating the mean. This includes using the $\bar{x} = \frac{\sum x}{n}$ formula or using an add-and-then-divide strategy if the symbolic representation of the formula is not yet learned. Other procedural knowledge not directly related to finding a value for the arithmetic mean, but necessary to further understand the mean includes the ability to algebraically manipulate formulae (e.g. $\bar{x} = \frac{\sum x_i}{n}$ can be rewritten as $n\bar{x} = \sum x_i$), and the ability to create graphs and tables (e.g. histograms and frequency distributions).

2.2.1.2 Mathematical Conceptual Knowledge of Arithmetic Mean

Fully understanding the mathematical concept of the arithmetic mean, ignoring for the moment any statistical role such as representativeness, involves the interrelationship between

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3 $$\bar{x} = \frac{\sum x_i}{n} \rightarrow n\bar{x} = \sum x_i \rightarrow \sum \bar{x} = \sum x_i \rightarrow \sum (\bar{x} - x_i) = 0$$

two areas of mathematics and the concepts and procedures within those areas. The arithmetic mean is rooted in arithmetic (addition, multiplication, and division) and algebra (manipulation of mean formula, mathematical properties of the formula, ratios, and properties of mathematical groups). The mathematics as it relates to the arithmetic mean of each of these areas is described below.

To understand the mathematical concept of mean it needs to be linked to knowledge of *arithmetic* operations. One such link is a conceptual understanding of addition and division. For example, $\bar{x} = \frac{\sum x_i}{n}$ means taking $\sum x_i$ and separating it into n equal parts of size $\bar{x}$. Furthermore, it is helpful to understand the connection between addition and multiplication, (i.e. $n\bar{x} = \sum x_i$ means that if you add $\bar{x}$ to itself n times you get the same total as summing the individual data points).

Portions of *algebra*, such as manipulating the arithmetic mean formula or finding missing variables within the formula, are key components to understanding the arithmetic mean. Knowledge of the algebraic ideas of ratios (see section 2.3.1.1) and proportions (see section 2.3.2.1) allows for the extension of mathematical ideas into other concepts related to the arithmetic mean, such as fair-share and center-of-balance. Mevarech (1983) found that the algebraic properties of a mathematical group play an important role in understanding (or misunderstanding) the arithmetic mean. The general binary operation of finding an arithmetic mean is not closed, does not follow the associative law, does not have an identity element, and its inverse is not its negation. The fact that these properties hold for the integers under addition and the rationals under multiplication (excluding zero), but fail in the computation of the mean, is a change in the existing schema for students. That is, the knowledge learned about the algebraic properties, either formally or informally, for addition and multiplication does not transfer to the algebraic properties of the arithmetic mean. For example, the binary operation of finding the arithmetic mean of two numbers, $\bar{a} = a * b = \frac{(a+b)}{2}$, does not have an identity element. That is, $(\bar{a} * y) \neq \bar{a}$ if y is different from $\bar{a}$. Unfortunately, students with limited experience working with algebraic properties tend to over generalize these group properties and their applicability (Mevarech, 1983). For example, an unsound understanding of the algebraic property of the identity leads to the misconception that zero can be added to a data set and the arithmetic mean will remain unchanged.

The connections formed within each mathematical area (i.e. arithmetic and algebra), in conjunction with the connections between these mathematical areas, form the network of knowledge or domain that shape mathematical understanding of the arithmetic mean.

2.2.2 Statistical Knowledge of the Arithmetic Mean

The statistical concept of average is characterized as a representative number that summarizes a data set (Russell & Mokros, 1996). Strauss and Bichler (1988) defined seven properties of the mean and categorized them into three aspects.[4] One of those aspects was *representative* and is defined by the property, "The average value is representative of the values that were averaged" (Strauss & Bichler, 1988, p.66). Unlike the other two aspects, which have mathematical elements, this aspect is strictly statistical in character because it is presenting a representative sense of the data. Pollatsek, Lima, and Well's (1981) describe the idea of representativeness as *functional* knowledge of the arithmetic mean, understanding "the mean is intended to be the quantity that best represents a set of scores" and "is an index of overall performance" (p. 199).

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[4] The three aspects are statistical, abstract, and representative. The representative aspect is discussed here; the statistical and abstract aspects will be discussed in later sections.

The arithmetic mean is a tool that describes a data set and, as such, allows comparison between data sets. The statistical nature of the arithmetic mean provides a means for drawing conclusions about the population or process from which the data originated (Cobb & Moore, 1997). The arithmetic mean is one of many averages that can alone, or in conjunction with other averages, be utilized to interpret the data. The statistical concept of the arithmetic mean utilizes a quantitative entity to represent, locate, qualify, describe, interpret, and/or signify a data set. A conceptual understanding of other areas of descriptive and inferential statistics could be valuable in further building a complete understanding of the arithmetic mean. For example, understanding the graphical representation of data could help students visualize the arithmetic mean; understanding appropriate experimental design and data collection could help students appreciate the effects of individual data points; and inferential qualities of statistics (e.g. confidence intervals) could help students realize the representative nature of the arithmetic mean. As with the mathematical conceptual knowledge of mean, the statistical conceptual knowledge of mean is rich in its relationships to other statistics, data representations, and conceptual ideas.

2.2.3 Connection between the Mathematical and Statistical Domains

The previous sections described the distinction between the mathematical concepts and statistical concept of the arithmetic mean. There is, however, a symbiotic relationship between the mathematics and statistics that contributes to the complete understanding of the arithmetic mean. To completely understand a multifaceted concept like the mean, one must understand all its dimensions and how they connect or interact. In the case of the arithmetic mean, which is "both central to statistical understanding and mathematically significant in a broader sense" (Mokros &

Russell, 1995), this means integrating its computation, mathematical relationships, and statistical aspects (Cai, 1998; Cobb & Moore, 1997).

As discussed earlier, Strauss and Bichler (1988) referred to three aspects related to the arithmetic mean. Two of these aspects, *abstract* and *statistical*, contain properties related to the both statistical and mathematical concepts and are provided with examples in Table 2-1 and Table 2-2, respectively.

Table 2-1: Abstract Aspects of the Arithmetic Mean

Property	Example
The average does not necessarily equal one of the values that was summed.	The mean of 10 and 20 is 15 (which is not one of the data points).
The average can be a fraction that has no counterpart in physical reality.	If an elementary school has two sections of first grade with 22 and 25 students in each classroom, then the mean number of students in each classroom is 23.5; a physical impossibility for an individual classroom.
When one calculates the average, a value of zero, if it appears, must be taken into account.	Five children were asked how many mathematics books they have in their homes. The responses were 1, 0, 4, 2, and 0. When calculating the mean the sum is not affected by the zero values (1+0+4+2+0 = 1+4+2), but zero values must be counted as part of the divisor (n=5).

Note: The properties are quoted directly from Strauss and Bichler (1988). The examples are the author's.

Table 2-2: Statistical Aspects of the Arithmetic Mean

Property	Explanation
The average is located between the extreme values.	The arithmetic mean can not be located above the highest value or below the lowest value, and in fact can not equal either value unless all data points are equal.
The sum of the deviations from the average is zero.	The sum of the differences of the mean subtracted from each data point is zero.
The average is influenced by values other than the average.	Any new data point added to the original data set will change the mean unless the new data point equals the mean

Note: The properties are quoted directly from Strauss and Bichler (1988). The explanations are the author's.

Fully comprehending each of these properties requires a conceptual understanding of both the mathematical and statistical nature of the arithmetic mean. The first two properties of the abstract aspect are mathematical in nature; they involve sums and fractions and are results of mathematical computations. The first two properties of the abstract aspect are also statistical in nature; they suggest the representative quality of the arithmetic mean. The third property of the abstract aspect is mathematical in nature due to the calculation involved and the identity property of zero with respect to addition. It is also statistical in nature due to the sense that the mean is representative of all data points, not just the non-zero data points. Each of the statistical aspects can be demonstrated using mathematical calculations or proofs and therefore have roots in the mathematical domain. Each of the statistical aspects also helps define the arithmetic mean as a representative number of a data set.

The connection of the statistical knowledge domain, the idea of representativeness, to mathematical knowledge is not a directly integrated relationship as each domain may reside independently in the general knowledge schema. Mayer and Greeno (1972) described how knowledge can be assimilated internally, such as when concepts are related in a domain, or externally, such as when concepts are related to ideas outside the domain. They investigated structural differences in the learning outcomes when two different methods of instruction were used to teach the binomial distribution. One method emphasized procedural calculation using the binomial formula while the second method emphasized general concepts and meaning of the variables. The research found that the way new ideas were assimilated into schema depended on the type of instruction to which each group was exposed. The group that focused on the formula connected the new knowledge to calculation techniques while the group exposed to the concepts of variables associated the newly learned knowledge into a more general conceptual sense. The authors explained their results by describing cognitive structures as consisting of both *internal* and *external* connectedness:

> An interpretation of the difference in terms of the learning outcomes achieved by the subjects can be developed by postulating two variables in cognitive structure. One is the extent to which components of a structure are integrated or connected with each other and could be called internal connectedness. The other variable is the extent to which the components of a structure are connected or related to other elements in a subject's general cognitive structure and this could be called external connectedness (Mayer & Greeno, 1972, p. 171).

Applying this idea to the knowledge of the arithmetic mean, the first of these variables, internal connectedness, can be related to the subset of mathematical procedures, along with their various mathematical concepts within the mathematical domain, necessary to understand the mean. The variable of external connectedness can be thought of as relating the mathematical and statistical domains of the arithmetic mean. The concepts of fair-share and center-of-balance may provide a cognitive bridge between the two domains. The concepts of fair-share and center-of-balance may be used to externally connect elements in the general cognitive structure, that is, mathematics to the statistical idea of representativeness. In this sense, the concepts of fair-share and center-of-balance represent Pollatsek, Lima, and Well's (1981) *analog* knowledge that translates between the mathematical and statistical descriptions of the arithmetic mean.

2.3 FAIR-SHARE AND CENTER-OF-BALANCE

In this section the concepts of fair-share and center-of balance will be described in three ways. First as individual concepts developed in the general knowledge schema, second as concepts related to procedures and concepts in mathematics, and third as concepts related to the statistical idea of representativeness.

2.3.1 Fair-Share

Fair-share is the equal partitioning of an object or equal distribution of objects to members of a group. Children learn the concept of fair-share through early social experiences, implicit ideals, and distributive counting. The action of partitioning objects or sharing is developed early in a child's development through experiences in social settings (Kieren, 1988). Initial sharing of objects begins with a rote understanding of "half" and often does not lead to equal partitioning (Pothier & Sawada, 1983). The most common and often earliest strategy for young children to clinically demonstrate the concept of fair sharing is dealing or systematically separating objects into groups without explicitly counting (Miller, 1984; Hunting & Sharply, 1988). Davis and Pitkethly (1990) found dealing is an implicit sharing strategy. Young children do not apparently have a conscious awareness that it is an adequate procedure for demonstrating equal sharing and (those that are capable) often resort to counting as a checking procedure. The concept of sharing which is evidenced early in a child's development by allocating pieces or objects in social settings or in counting activities leads to an intuitive sense of fair-share.

2.3.1.1 Fair-Share as a Mathematical Concept

Fischbein, Deri, Nello, and Marino (1985) hypothesized that, "Each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model" (p. 4). Fair-share is the intuitive model that underlies partitive division. Partitive division is characterized by an object or collection of objects being divided or distributed into equal portions or an equal number of subcollections. In its most basic sense, the arithmetic mean is calculated by equally dividing a set of objects into a given number of subsets. Therefore, the arithmetic mean formula is an example of partitive division.

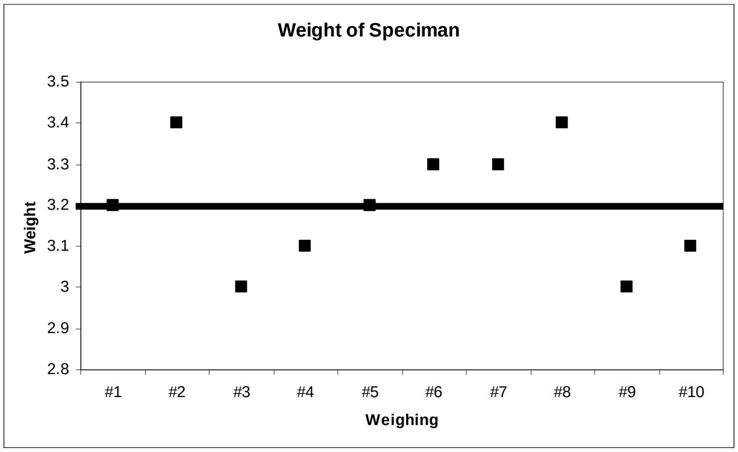
Another description of the mean that utilizes the notion of fair-share is as a normalized ratio (Cortina, 2002). In this case, fair-share can be used to construct units of measure (e.g. miles per gallon) that do not necessarily involve statistical variability or prediction but have mathematical relationships. In this multiplicative conceptualization, the mean is an attribute of a group of data points in which an aggregate measure is created by summing all of the individual data point values. If one divides this aggregate measure by the total number of individual measures that created it, a normalizing adjustment, a "group performance relative to the number of individual contributors," is produced. "In this sense the mean is like an average rate; the measure of the group contributions per contributor is conceived to be the same as the amount contributed by each *n* contributors if each were to contribute equal amounts" (Cortina, Saldanha

& Thompson, 1999, p. 2). This is equivalent to the group total being equally distributed (i.e. shared fairly) amongst the contributors.

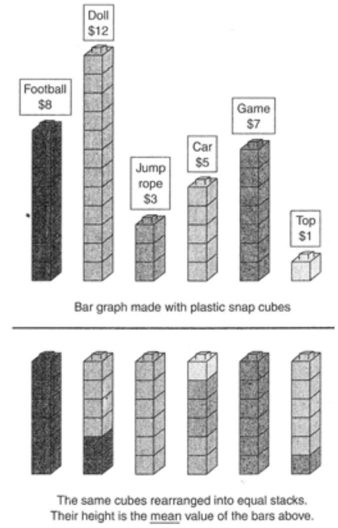
2.3.1.2 Fair-Share as a Statistical Concept

The arithmetic mean can be statistically conceived as representative by a fair-share model. “The basic question underlying the fair share model is, what would be an equal share if all items were distributed [equally]?” (Mokros & Russell, 1995, p. 21). The conceptualization of fair-share can be illustrated using a *signal-in-a-noisy-process* or *equal-redistribution* idea. Konold and Pollatsek (2002) argue that the arithmetic mean can be thought of as a signature-signal and the data can be thought of as a noisy-process. For example, the observed weights of an object on a scale may vary each time the object is weighed. The variations in the measurements can be thought of as a noisy-process. The true weight can be estimated by an arithmetic mean or signal that levels off the variation (Groth, 2005). An illustration of this approach is found in Figure 2-1(a).

A second conceptualization of the arithmetic mean as a fair-share is the equal-redistribution model. The data values are represented as block columns with the height of the columns equaling the value of each data point. Redistributing the blocks such that all columns are the same height equates to allocating the total sum in the data set equally amongst each data point. The new height of equal columns, or evenly distributed columns, is equivalent to the mean of the data set. (Van DeWalle, 2003). This process is depicted in Figure 2-1(b).



(a)



(b)

(Van de Walle & Lovin, 2006)

Figure 2-1: The Mean as a Fair Share

These models help depict the concept of fair-share as it relates to the statistical concept of representativeness of the arithmetic mean. Each fair-share model is an interpretation of the arithmetic mean as a location, description, and/or representation of the mean in relation to the data set.

2.3.2 Center-of-Balance

The concept of center-of-balance has been widely studied to mark cognitive development in children since Inhelder and Piaget (1958). A balance scale, a device in which weights can be added to each side of a lever arm that pivots on a fulcrum, offers a diverse sequence of rules through which children progress. Siegler (1976) proposed four rules to track cognitive development:

Rule I: Four and five-year-olds base predictions only on the relative weight on each side of the fulcrum.

Rule II: Eight and nine-year-olds consider distance from the fulcrum if the weight on each side is equal but rely only on weight if the weights differ.

Rule III: Twelve and thirteen-year-olds consider both weight and distance, but do not know how to resolve conflicts if weights and distances differ.

Rule IV: Few children and adults rely on torques (i.e. multiplying weight by its distance).

Experiments performed by Lovell (1961) and Jackson (1965), and later confirmed by Siegler (1976), indicated that only 20% of adults are capable of using Rule IV for center-of-balance problems; Siegler termed performance at this level as mature. Furthermore, Hardiman, Pollatsek, and Well (1986) report, "Even when provided with specific experiences intended to promote understanding of the concept of balance, adults do not easily derive the product-moment rule [torque]" (p. 64).

The concept of center-of-balance has been widely studied and its development is well documented. The concept develops in complexity of understanding through Rules I – III with age and experience, but does not progress to mature performance (Rule IV) without explicit instruction.

2.3.2.1 Center-of-Balance as a Mathematical Concept

The concept of center-of-balance has direct connections to procedures and concepts in mathematics. Torque (Rule IV) is calculated using multiplication, addition, and vector cross products. Hardiman, et al. (1986) define the mathematical calculation of center-of-balance as,

> The effectiveness of a weight in causing the [balance] beam to tip is determined by the product of the weight (*w*) and its distance from the fulcrum (*d*), a construct called

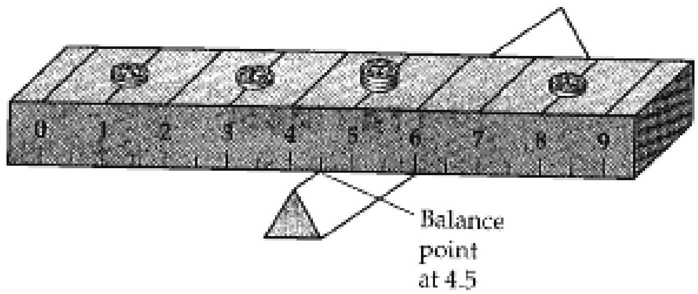
torque….If the total torque (i.e., $\sum w_i d_i$) associated with the weights on each side of the beam is the same, the beam will balance, otherwise the beam will tip to the side with the greater torque. (p.64)

Siegler (1976) states, “It [balancing] is an interesting task mathematically, being related to the concept of proportionality” (p. 482). Another important link between center-of-balance and mathematics is the notion of equality and its role in balancing weights and torques in equations that represent data sets and their deviations from the mean.

2.3.2.2 Center-of-Balance as a Statistical Concept

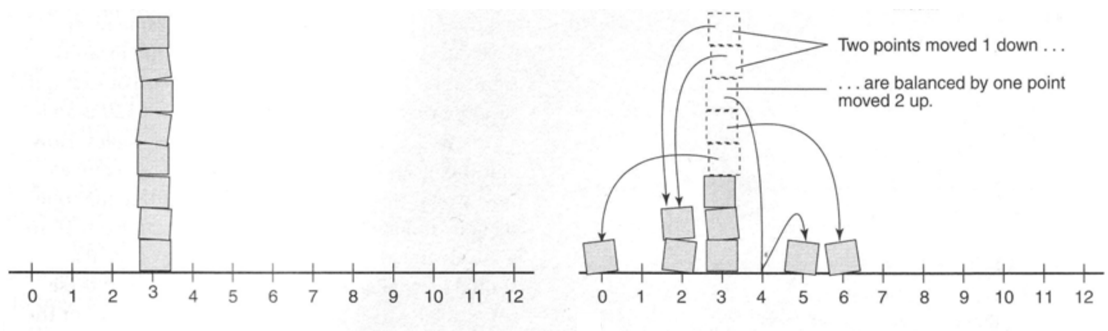
The arithmetic mean can be statistically conceived as representative by a center-of-balance model. Figure 2-2 shows how the representative nature can be demonstrated by a “balance” diagram or by a “block-stacking” procedure depicting center-of-balance. The top picture (a) depicts the arithmetic mean as the balancing point of a scale in which the frequency distribution of the data points is analogous to the distribution of the weights. The fulcrum can be thought of as the point that represents the entire data set. A second approach, presented as (b) in the figure, uses a manipulative to build a column of blocks for each value in the data set. The blocks are moved toward the center of the distribution, carefully moving an equal number of blocks an equal distance from each side of the hypothesized center, in order to keep the model balanced. The point on which all blocks can be stacked after equal movement is the arithmetic mean. It can be thought of as the point that best represents the original distribution of the data set.

(a)



(Aufmann, Lockwood, Nation, & Clegg, 2007)

(b)



(Van de Walle & Lovin, 2006)

Figure 2-2: The Mean as a Center of Balance

The balancing models help illustrate how the center-of-balance concept is related to the idea of representativeness. "This conception [balance beam] should be useful because it allows connections to be made to general knowledge of and experience with balancing, leads to reasonably accurate approximations to the mean and helps make clear that it's the relative frequencies of scores that are important in determining the mean (Hardiman, Well, & Pollatsek, 1984, p. 794). The center-of-balance models describe the arithmetic mean as a location and/or representation relevant to the data.

2.3.3 Relationship between Fair-Share and Center-of-Balance

The relationship between the fair-share and center-of-balance conceptualizations is a focus of this study (research question #1). Of particular interest are how the differences in the two conceptualizations can be cognitively integrated and how this integration manages to describe the statistical and mathematical nature of the arithmetic mean.

Figure 2-3 illustrates some conceptual differences for several mathematical contexts of fair-share and center-of-balance in relation to the arithmetic mean.

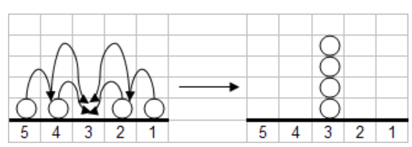
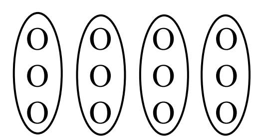
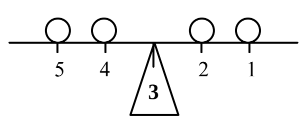
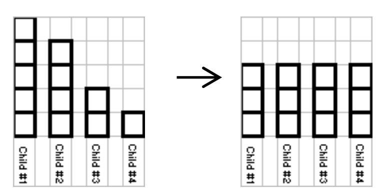
	Center-of-Balance	**Fair-Share**
Real World Situation	Four children bring 5, 4, 2, 1 M&M's to school. What is the mean number of M&M's?	Four children bring 5, 4, 2, 1 M&M's to school. How can the M&M's be fairly shared?
Spoken Symbol	"I find the number that is in the middle or balances the number of M&M's each child brought"	"I put all the M&M's in a pile and divvy-them-up equally"
Pictures		
Manipulative Model		
Written Symbol	$(5-x) + (4-x) + (2-x) + (1-x) = 0$ $12 - 4x = 0 \ldots x = 3$	$5 + 4 + 2 + 1 = 12 \quad \ldots \quad 12 \div 4 = 3$

Figure 2-3: Examples of Differences in Center-of-Balance and Fair-Share

Most previous research on the arithmetic mean's relationship to fair-share and center-of-balance reports on the effects each conceptualization has on student understanding of the arithmetic mean. While the internal connections between and within the knowledge elements of the mathematical domain have been methodically studied (see e.g., Gray, Pinto, Pitta, & Tall, 1999 for between; Lesh, Post & Behr, 1987 for within); currently, little research has focused on the links between the different conceptualizations, or the external connection between the conceptualizations of fair-share and center-of-balance.

MacCullough (2007) studied how experts[5] understand the arithmetic mean. She used task-based interviews to determine how subjects understood and related specific problems associated with the arithmetic mean. Based on her results she hypothesized how a leveling-off strategy could connect the notions of fair-share and center-of-balance.

> The experts understood the algorithm for arithmetic mean as a result of partitive division. The data values were accumulated and then shared fairly with each data point. This was equated to leveling-off by suggesting that the fair sharing could be done by simply moving pieces of a bar graph (or numerical amounts) until every data point was the same value. The activity of leveling-off allowed the experts to "visualize" the deviations from a proposed mean. In order to obtain balance, and thus find the arithmetic mean, the pieces over and the pieces under had to be equivalent. The experts implied that leveling-off would find a point of balance because any amount over the mean exactly matched an amount under the mean. In this sense, the leveling-off was the same as numerically

[5] The experts were: a mathematics educator; a graduate student; a statistics educator, a statistician, and a mechanical engineer

> cancelling deviations. When leveling-off to find a point of balance, the experts focused on the deviations from the mean and their equivalence (p. 99).

This explanation of the external connection between fair-share and center-of-balance relies on two key mathematical concepts. The first, partitive division, is a general mathematical concept rooted in the conceptualization of fair-share. The second key concept is the 'sum of the deviations from the mean is zero.' In this case, the concept is being related to the of center-of-balance conceptualization. The relationship between fair-share and center-of-balance to these two concepts may indicate the external connection between fair-share and center-of-balance is connected through either the statistical or mathematical domains.

2.4 A KNOWLEDGE STRUCTURE FOR THE ARITHMETIC MEAN

The research on the development of knowledge for the arithmetic mean (see section 2.1) and the conceptualizations of fair-share and center-of-balance (see section 2.3) that relate mathematical and statistical domains (see section 2.2) can be combined to form a hypothesized structure of how knowledge is related for the arithmetic mean. This is shown in Figure 2-4. The structure has two knowledge domains, mathematical and statistical. Within the mathematical domain there are procedures and concepts related to mathematical calculation of the arithmetic mean. The statistical domain is characterized by the concept of representativeness and how the statistical context of the data is represented by arithmetic mean. Section 2.3.1 and section 2.3.2 described the connections between the statistical domain and each of the conceptualizations (i.e. fair-share and center-of-balance) and the connections between the mathematical domain and each of the conceptualizations, respectively. It is hypothesized that fair-share and center-of-balance conceptualizations bridge the statistical concept of representativeness to the seemingly unrelated mathematical procedures and concepts for calculating the arithmetic mean. The conceptualizations of fair-share and center-of-balance may function as cognitive bridges between the statistical and mathematical domains. As described in section 2.3.3, a focus of this study is the nature of the cognitive connection (if any exists) between fair-share and center-of-balance conceptualizations of the arithmetic mean.

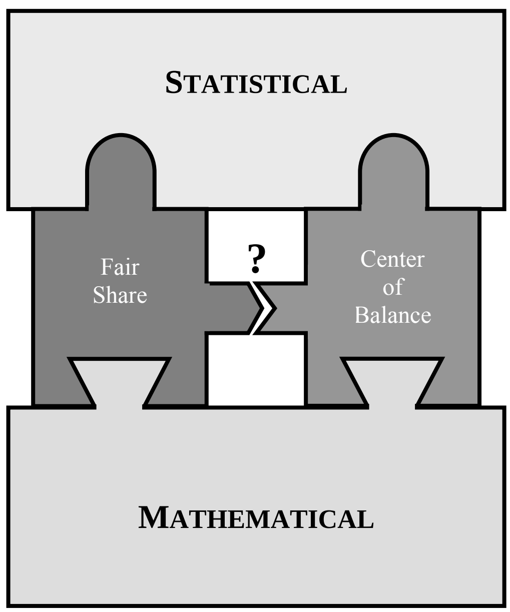


Figure 2-4: Knowledge Structure of the Arithmetic Mean

Based on previous research it can be hypothesized that knowledge within the mathematical domain is internally connected and most likely grows in an iterative fashion combining procedural and conceptual knowledge (Ambrose, Baek, & Carpenter, 2003; Rittle-

Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001). The knowledge relating the statistical and mathematical domains is conceptually integrated across the disciplines and may be externally connected. How students connect the concepts of fair-share and center-of-balance to the mathematical domain is of interest to this study (research question #2). The following sections describe how relationships between disciplines or domains of knowledge interact and inform each other.

2.4.1 Relating Mathematical Concepts to Concepts in Other Disciplines

Mathematics itself is a dynamic discipline with a historic foundation and a seemingly infinite space for growth; it also contributes to society as a tool in numerous ancillary disciplines. Fields of study such as the physical sciences, economics, engineering, computer science, and statistics, to name a few, heavily utilize mathematics within their discipline. Ideally, the relationship of knowledge between mathematics and other disciplines should be conceptually integrated rather than consisting of borrowed procedures.

In experimental physics, for example, mathematics plays an indirect procedural role of empirically determining facts or providing a conceptual basis for physical understanding. For example, a mathematical understanding of calculus helps explain the conceptual relationship between acceleration, velocity, and position in mechanics. Sauer (2000) found students who used mathematical modeling to conceptually construct formulas had a more flexible approach and a more conceptually correct view of acceleration than did students who were given the formulas. Sherin (2001) proposed the following explanation of the relationship between mathematical concepts and concepts in physics:

> The use of formal expressions in physics is not just a matter of the rigorous and routinized application of principles, followed by the formal manipulation of expressions to obtain an answer. Rather, successful students learn to understand what equations say in a fundamental sense; they have a feel for expressions, and this guides their work. More specifically, students learn to understand physics equations in terms of a vocabulary of elements that I call *symbolic forms*. Each symbolic form associates a simple conceptual schema with a pattern of symbols in an equation. From the point of view of improving instruction, it is absolutely critical to acknowledge that physics expertise involves this more flexible and generative understanding of equations, and our instruction should be geared toward helping students acquire this understanding (p. 479).

A study that analyzed videotapes of students solving physics problems found students with a contextual (physics) understanding of an equation do not only use the equation at the start of the problem, but throughout the solution process, allowing for more flexible problem solving. This indicates the relationship between mathematical and physics concepts is iterative during the problem solving process (Sherin, 2001).

In statistics, mathematics is often used to assist in solving a problem, but only after considerable statistical thinking and reasoning have been accomplished (delMas, 2004). DelMas contends that students unable to relate statistical and mathematical reasoning resort to solutions based on "the output of associative processes that fall short of the reflection and integration needed for complete understanding" (p. 90). Solutions based solely on the output of associative processes, such as procedurally computing the arithmetic mean using the mean formula, are often unable to be related to statistical concepts, such as representativeness. On the other hand, if the associative process is linked to a concept, and that concept is linked to a statistical idea, then a conceptual relationship can be built between the associative process and the statistical idea. For example, Cortina (2002) found students improved their understanding of the arithmetic mean when they associated the arithmetic mean formula with a ratio representing a per-one contribution to the aggregate total of the data, a fair-share notion. As previously discussed, the fair-share notion is linked to the statistical idea of representativeness. The linking of the arithmetic mean formula to the notion of fair-share, followed by connecting fair-share to the idea of representativeness, cognitively connects the mathematical notion of the arithmetic mean formula to the statistical idea of representativeness. A caveat, Hardiman, Well, and Pollatsek (1984) found using a balance beam analogy to represent the arithmetic mean improved a student's understanding of the mean, but only to the degree that the center-of-balance model and arithmetic mean formula were conceptually understood as separate entities. Thus, the learning that delMas describes and Cortina demonstrated is limited by the understanding of concepts within the domains and cognitive bridges; not just by the understanding of their connection. Therefore, the concepts inherent in the domains and cognitive bridges themselves need to be developed for optimal understanding to emerge.

The research studies presented above indicate a more developed conceptual understanding of the mathematical knowledge related to complementary fields leads to an increased conceptual understanding of the knowledge within the related field (e.g. physics and statistics). As with the iterative relationship between procedural and conceptual knowledge (Ambrose, Baek, and Carpenter, 2003; Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001), it can be hypothesized that not only does the related domain knowledge coalesce as conceptual understanding of the mathematics increases, but the contextual representation of the mathematics within the related domain could strengthen the conceptual understanding of the mathematics.

2.4.1.1 Cognitive Blending

Fauconnier and Turner (1998) have described the connections that conceptually integrate domains for analogy or metaphor as *cognitive blending*. In cognitive blending, inputs from each domain are connected in a blended space at various levels of abstraction to create an emergent structure not directly available from the input domains alone. "A particular process of meaning construction has particular input representations; during the process, inferences, emotions, and event-integrations emerge which cannot reside in any of the inputs; they have been constructed dynamically in a new mental space—the blended space—linked to inputs in systematic ways" (Fauconnier & Turner, 1998, p. 135).

Bing and Redish (2007) provide examples of how cognitive blending can be used to explain the connection between mathematical concepts and physics concepts. They analyzed data from video tapes of physics and engineering mechanics students completing homework problems. Evidence suggests students combined particular concepts in mathematics with physical concepts to produce a blended space describing the concept of air drag. Figure 2-5 is a depiction of a cognitive blend between the mathematics and physics domains for the concept of air drag.

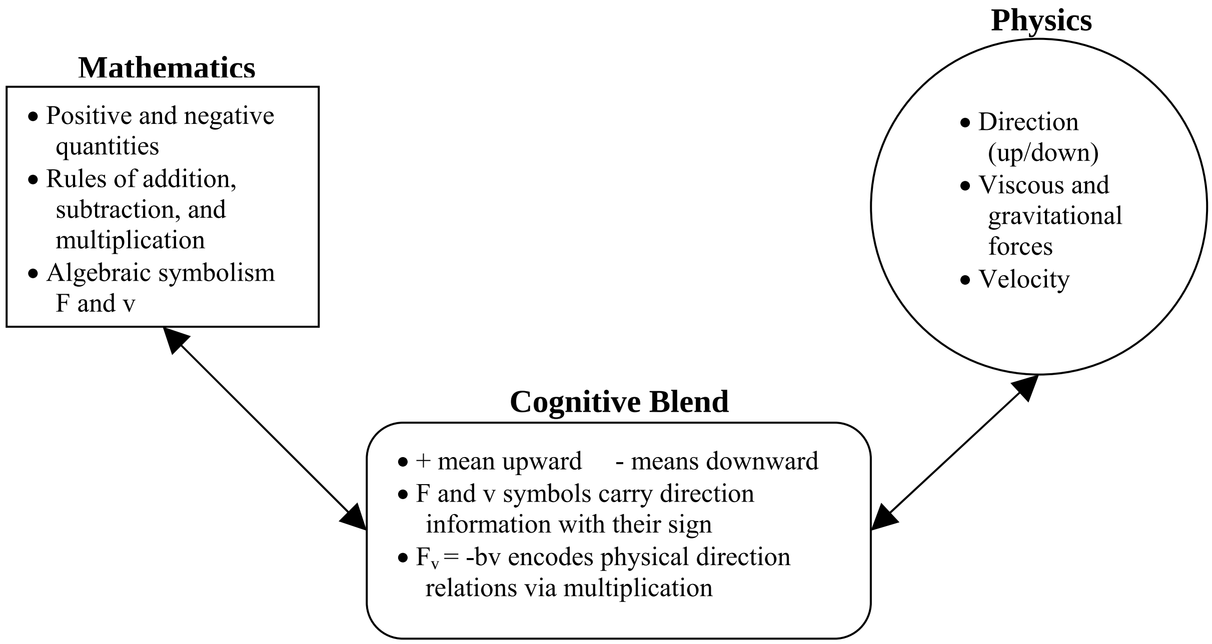


Figure 2-5: Mathematics/Physics Blending Example

The mental space provided by the cognitive blend provides a means to achieve the complex integration of mathematical and physical concepts. Bing and Redish (2007) found difficulties by students in understanding the concept of air drag were not from lack of prerequisite knowledge of mathematics or physics, but from an inappropriate integration of two well-established mental spaces. They concluded that "awareness of a cognitive blending framework can help instructors more readily understand how students are thinking and offer appropriate guidance for the situation at hand" and "help researchers in providing a theoretical framework for description of student thought and perhaps even a structure for understanding what cues prompt students for blending in particular ways" (p. 29). Therefore a better understanding of the blending spaces or providing mental spaces for blending to occur can help in instruction and student understanding.

The next section relates how the cognitive bridges of fair-share and center-of-balance might be used as blending spaces for the mathematical and statistical domains of the arithmetic mean. Inputs from each domain, internally linked to the concepts of fair-share and center-of-balance, are dynamically blended enabling a more complete understanding of the arithmetic mean.

2.4.2 Using Fair-Share and Center-of-Balance for Connecting Domains

As discussed is section 2.2.3, fully comprehending all aspects of the arithmetic mean involves connecting knowledge from mathematical and statistical domains. The conceptualizations of fair-share and center-of-balance may provide cognitive spaces to blend procedures and concepts from the mathematical and statistical domains. In the case of the arithmetic mean, the cognitive spaces for blending the two domains are themselves well defined concepts (i.e. fair-share and center-of-balance). This is different than the blending spaces the physics students in Bing and Redish's (2007) study specifically constructed between the physics and mathematics domains. The advantage (or disadvantage) of using a well defined cognitive blending space versus a constructed space has not yet been reported in current research. Results from this study may contribute to this discourse. An example of how the concept of center-of-balance might be used to cognitively blend knowledge from the mathematical and statistical domains is demonstrated in Figure 2-6.

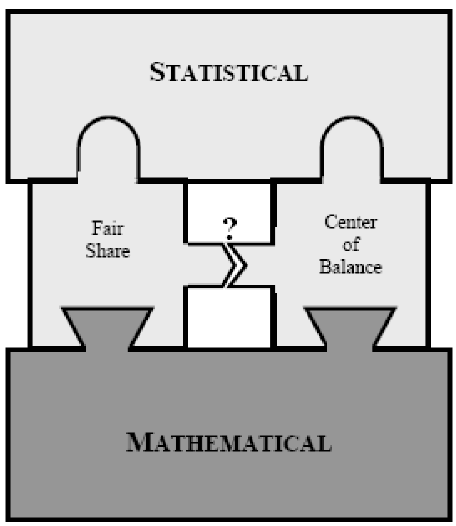
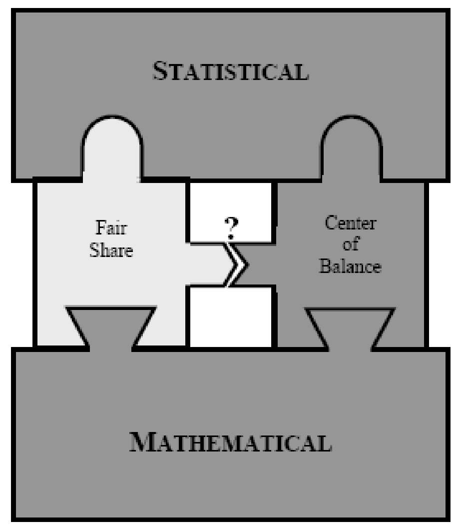
Goal:	Relate the arithmetic mean formula to the arithmetic mean property: the sum of the deviations from the mean is zero	
Method:	Algebraically manipulate the arithmetic mean formula to represent the property	Use the notion of center-of-balance and the block-stacking technique to demonstrate the property
Actions:	$\bar{x} = \frac{\sum x_i}{n} \rightarrow$ $n\bar{x} = \sum x_i \rightarrow$ $\sum \bar{x} = \sum x_i \rightarrow$ $\sum (\bar{x} - x_i) = 0$	1) The arithmetic mean formula (or an example that utilizes it) can be depicted as a block stacking diagram 2) The blocking stacking technique portrays the notion of center of balance 3) Center of balance can help conceptualize the sum of the deviations from the mean is zero
Accessed Knowledge:		

Figure 2-6: Interactions of Internal/External Connections

The examples in Figure 2-6 represent plausible conceptualizations of the arithmetic mean property, ‘the sum of the deviations from the mean is zero’ (Strauss and Bichler, 1988). The first example utilizes an algebraic manipulation of the arithmetic mean formula to demonstrate that the property is inherent in the formula. All of the knowledge necessary to complete the calculation resides in the mathematical domain. There is no evidence that any idea of representativeness would be apparent to a student exposed to this derivation. The second example utilizes the block-stacking technique and the notion of center-of-balance to build the representative sense of the property; 'the sum of the deviations from the mean is zero.' By utilizing the concept of center-of-balance to blend mathematical and statistical domains, the arithmetic mean property has an opportunity to be conceptualized and connected to both domains as well as to the concept of center-of-balance.

A better understanding of the relationship and connection between the blending spaces of fair-share and center-of-balance, and how they relate to the relevant mathematics, could help in pedagogically presenting a comprehensive picture of the arithmetic mean. The following section indicates how incorporating the concepts of fair-share and center-of-balance into instruction of the arithmetic mean increases conceptual understanding and reinforces the statistical concept of representativeness.

2.5 INSTRUCTIONAL INVESTIGATIONS

A goal of research regarding the arithmetic mean is to improve students' understanding. A necessary part of improving understanding is to improve the teaching of the arithmetic mean; this entails advancing teacher understanding, teaching techniques, instructional materials, and assessment. Improvement in statistics education parallels current efforts of reform in mathematics education that focus on problem solving, conceptual understanding, and technology (Garfield, 1995; Greer, 2000). Statistics also provides a vehicle to contextualize mathematics into "real-world" situations.

Important to instruction of the arithmetic mean is presenting situations and problems that contextually help the students conceptualize it. "The continual shuttling backwards and forwards between thinking in the context sphere and statistical sphere" utilizes statistical thinking that an expert statistician would use, and also contextually bases the results and conclusions (Wild & Pfannkuch, 1999, p.228). This contextualization was an essential part of a study by McClain and Cobb (2001) that found middle school students "reconceptualized their understanding of what it means to know and do statistics" (p. 126) when given the opportunity to explain and justify their answers within the context of the problem.

Students tend to think of the arithmetic mean as a procedure rather than a tool for data analysis (McGatha, Cobb, & McClain, 2002). To remedy this, the researchers suggest that instruction treat data analysis as an inquiry rather than as a procedure. "In cases where there is a conflict between intuitive estimates and formal measures, they [students] should be encouraged to find causes of that conflict rather than simply replacing their intuitive estimates with formal tools that produce 'correct' school-sanctioned answers" (Groth, 2005, p. 14). Students develop a more conceptually based notion of the arithmetic mean if they are permitted to make many informal estimates based on their intuitive notion of average rather than use a formal measure, such as the arithmetic mean formula, to calculate an answer.

Previous research indicates the nature in which a teacher presents the arithmetic mean, mathematically and/or statistically, and the focus of connection between the mathematical and statistical domains, fair-share or center-of-balance, impacts the outcome the instruction. Researchers such as Mokros and Russell (1995), Pollatsek, Lima, & Well (1981), and Strauss (1987) emphasized the mean as a balancing-point or fair-share. The conceptualization of center-of-balance can be modeled as a balance diagram or block-stacking while the conceptualization of fair-share can be modeled as either a signal-in-a-noisy process or block-leveling as previously discussed (see sections 2.3.1.2 and 2.3.2.2). Hardiman et al. (1984) performed an experimental study in which students in two groups were given either instruction on a balance model or were placed in a control group after having solved several mean and weighted mean problems on a pretest. The results indicated the balance model instruction "led to better understanding of the weighted mean" (p.799). The equal-redistribution model has also been linked to student learning. Cai and Moyer (1995) found students use three different mathematical representations, verbal (written words), symbolic (mathematical expressions), and pictorial (drawings) when solving problems related to the arithmetic mean. The representations that students utilized appeared to be directly related to their chosen solution strategy. For example, students that used an add-and-then-divide strategy most often utilized a symbolic representation of the arithmetic mean formula. Students that used a leveling strategy most often utilized a pictorial representation. Students that exhibited a higher conceptual understanding of the arithmetic mean often used multiple representation strategies in their solutions. The results of the study showed an increase in the conceptual understanding of the mean after the open-ended problem solving instruction. George (1995) found that students exposed to a reform curriculum that encouraged constructing knowledge exhibited a better conceptual understanding of the mean than students who were taught the add-and-then-divide procedure. In this case, the reform curriculum was *Visual Mathematics* which introduced the mean using the equal-redistribution model. A study presenting the fair-share conceptualization of the arithmetic mean as a signal-in-a-noisy-process found several students utilized higher level statistical thinking in contextualizing arithmetic mean problems using this model (Groth, 2005).

In conclusion, research studies have indicated that students' misconceptions regarding the mean are strong and resilient. They are not easily changed even if faced with contrary evidence (Garfield, 1995). Effective instructional practices related to the arithmetic mean mimic the qualities and activities of reform mathematics teaching. Instruction that contextualizes problems and relates the arithmetic mean to specific statistical models has led to improved student understanding.

One question in statistics education that previous research has not answered is which conceptualization, fair-share or center-of-balance, and which model (e.g. signal-in-a-noisy-process, redistribution of blocks, balance beam, block-stacking) related to these concepts, is most effective in connecting the representative nature of the arithmetic mean to its mathematical constructs. The study described here is meant to resolve this question.

2.6 SUMMARY AND IMPLICATIONS OF LITERATURE

Among averages, the arithmetic mean is unique in that the conceptual basis from which it is developed, representativeness, is not typically developed before the procedure to calculate it is introduced. Without connections to the statistically founded concept of representativeness, one's knowledge of the arithmetic mean seems limited to computation of the mathematical formula; thus allowing little or no access to mathematically and statistically rich or adaptive problems, including those that arise in our everyday lives.

Statisticians contend that statistics is not a subfield of mathematics, but rather its own field that utilizes mathematics, much like physics or economics. A statistician understands the concepts of statistics and the significance of statistical thinking and uses the tools of mathematics to solve or predict within the context of a problem. This idea suggests the need to develop a statistical sense of the arithmetic mean before a procedural technique is introduced.

Several studies have researched the type of knowledge, procedural or conceptual, that students of different age groups utilize to solve arithmetic mean problems (Cai & Moyer, 1995; Cai, Moyer & Grochowski, 1999; George, 1995; Groth, 2005; Hardiman et al. 1984). Research in mathematics education indicates these two types of knowledge work together to form a complete understanding of a particular topic. In terms of the arithmetic mean, procedural and conceptual knowledge may appear to be a dichotomy, with procedural knowledge hindering the conceptual understanding of the mean. A possible explanation to this inconsistency is that the statistical conceptual knowledge of the arithmetic mean resides in a different domain than the procedural and conceptual knowledge that shape its mathematical constructs. The notion of cognitive blending may offer insight as to the growth of knowledge with respect to the arithmetic mean.

The concepts of fair-share and center-of-balance offer a cognitive space for the blending of the mathematical and statistical domains of knowledge intrinsic to the arithmetic mean. Research has indicated how each concept is individually related to both the mathematical and statistical domains, but offers little as to how the concepts of fair-share and center-of-balance are cognitively related to each other. It is also unclear as to how understanding of statistical concepts enhances the mathematical knowledge related to the arithmetic mean, and conversely, how mathematical knowledge shapes the understanding of statistical concepts. While problem solving skills and understanding of the arithmetic mean increase with age, it is uncertain if this growth is due to a better understanding of mathematical knowledge, an increase in the understanding of the statistical concepts, or a combination of both. A better understanding of how mathematical and statistical knowledge relate, interact, and grow could lead to improved pedagogy of the arithmetic mean.

3.0 METHODOLOGY

The goal of this study is to identify and describe the cognitive relationship between the concepts of fair-share and center-of-balance, as well as the cognitive relationship between these conceptualizations and the mathematical domain. These relationships were described in two ways: (a) the extent knowledge of cognitive blending spaces, such as fair-share and center-of-balance, affect each other and/or affect knowledge of mathematical concepts related to the arithmetic mean; and (b) the nature of the cognitive relationships that exist between the conceptualizations of fair-share and center-of balance, and between these conceptualizations and the mathematical domain.

This chapter begins with a general discussion of the research design for this study. Second, the chapter details the demographics of the participants involved in the research study. Third, aspects of data collection, data coding, and data analysis are systematically outlined. Data collection includes the specifics of the instrument and data gathering procedure. The section concerning data coding reveals the mechanisms of the rubric utilized to quantify the data and the coding scheme used to qualify the data. The final section dealing with the data, data analysis, details the statistical testing and qualitative examination of the data. Results from a pilot study that influenced the collection, coding, and analysis of the data are presented throughout the chapter.

3.1 RESEARCH DESIGN

Examining the thought and solution processes of individuals as they solve problems is an effective method for eliciting understanding about their mathematical knowledge. Mathematical knowledge is constructed from cognitive relationships among concepts that exist in an individual's knowledge schema. Using instructional interventions can impact an individual's mathematical knowledge by manipulating, reinforcing, or increasing existing cognitive relationships in their knowledge schema. Comparing thought and solution processes before and after instructional interventions can illustrate the impact the new knowledge has on the existing knowledge schema. The changes, if any, in the knowledge schema represent the cognitive relationships between the induced knowledge of the knowledge intervention and the existing knowledge.

This study utilized a pre- and post- test design of randomly assigned participants belonging to one of three groups: (a) those given access to instruction on fair-share knowledge, (b) those given access to instruction on center-of-balance knowledge, and (c) a control group receiving instruction on general problem solving heuristics. The methodology of protocol analysis was used to gather, code, and analyze the data. Data collection consisted of think-aloud verbal protocols gathered from pre- and post- tests of arithmetic mean problem solving sessions with an instructional intervention between the two test administrations. The data was coded for evidence of knowledge of different domains (i.e. mathematical or statistical) and different conceptualizations (i.e. fair-share and center-of-balance) as proposed in the knowledge structure for the arithmetic mean (see section 2.4). The coded data was analyzed using two different, but related schemes. The first scheme used statistical analysis to quantitatively locate any significant relationships between the concepts of fair-share and center-of-balance; along with any connections between these and mathematical concepts. The second scheme included qualitatively analyzing the data to explore any cognitive relationships identified in the initial statistical analysis, or any cognitive relationships apparent in the verbal protocols but not detected by the statistical analysis or in the initial pilot study. This two-scheme mixed-method approach helped ensure thorough analysis of the data.

Mixed-method methodologies, or combining quantitative and qualitative methods, have been used by researchers to deepen the insights from and expand the scope of their studies (Sandelowski, 2000). Chi (1997) explains the rational for using a mixed methodology, particularly in the case where verbal protocols are used:

> There are clearly many advantages and shortcomings to both qualitative and quantitative methods. The main advantage of qualitative research is that it can provide a richer and deeper understanding of a situation. Moreover,…many skills are executed in a very different way in context than in a sterile laboratory environment. However, qualitative methods usually suffer form subjective interpretation and nonreplicability. Quantitative methods, on the other hand, have the advantage of objectivity and replicability, but the shortcoming is that one can only make conclusions about the specific hypothesis at hand. Furthermore, the sterile laboratory environment of experimental studies limits the generalization of the results to a real-world context. Clearly, there is a need to blend the two methods in such a way to remove each method’s shortcomings. The verbal analysis method attempts to satisfy these goals by removing subjectivity and yet maintaining the richness of context (p. 280).

There are three primary reasons for integrating quantitative and qualitative methods in sampling, data collection, data coding, and/or data analysis (Greene, Caracelli, & Graham, 1989):

1) Interpretation – using qualitative data to help interpret, clarify, explain, or otherwise more fully elaborate the results of quantitative analysis

2) Confirmation – treating qualitative and quantitative data with equal weights to achieve or ensure corroboration of data or convergent validation

3) Development – using results from qualitative analysis to generate a hypothesis or guide the use of additional sampling, data collection, and analysis techniques that will be tested using quantitative methods

The mixed-methodology of this study was applied at the data coding and data analysis stages. The data, transcripts of problem solving verbalizations and corresponding written solutions, were quantified using a rubric for statistical analysis and qualified using a coding scheme that identified knowledge usage. Verbal protocols with written artifacts were utilized to interpret, clarify, explain, or otherwise more fully elaborate the results of the statistical analyses. The verbal protocols and written artifacts supplied insight into the cognitive processes suggested by the statistical analyses.

The quantitative and qualitative analyses were combined to answer the following research questions:

1) How is knowledge of fair-share and center-of-balance cognitively related to one another? In particular,

   a) What effect does instruction of the fair-share conceptualization of the arithmetic mean have on knowledge of the center-of-balance conceptualization?

   b) What effect does instruction of the center-of-balance conceptualization of the arithmetic mean have on knowledge of the fair-share conceptualization?

2) How is knowledge of fair-share and center-of-balance cognitively related to the mathematical domain? In particular,

a) What effect does instruction of the fair-share conceptualization of the arithmetic mean have on knowledge of mathematical concepts associated with the arithmetic mean?

b) What effect does instruction of the center-of-balance conceptualization of the arithmetic mean have on knowledge of mathematical concepts associated with the arithmetic mean?

3.2 PARTICIPANTS

Undergraduate liberal arts students tend to generate a broad range of mathematical and statistical knowledge. A breadth of knowledge regarding the arithmetic mean was necessary in order to answer the research questions in this study. The chosen sample population of participants was enrolled in a liberal arts mathematics course at a small private university. The first term of the course fulfilled the only required mathematics credits for most students enrolled (education majors are required to take a second term). Students take the course based on their major (e.g. education, journalism, humanities, and performing arts). Their mathematical and statistical backgrounds varied depending on the level of mathematics achieved at the high-school level. By design, none of the participants had previously taken a formal statistics course. One part of the liberal arts mathematics course in which the students were enrolled is devoted to statistics, and, in particular, one lecture focuses on measures of central tendency including the arithmetic mean; that section, however, was not yet covered at the time of this study.

Sixty potential participants for the proposed study were enrolled in three sections of the liberal arts mathematics course. Of that initial group, thirty-eight agreed to participate in the research study, and thirty were randomly selected to partake in the study. Each participant was given a participant-number alias to identify him or her during data coding and analysis.

Previous qualitative research studies attempting to gain insight about knowledge of the arithmetic mean have utilized similar sample sizes. These include Pollatsek, Lima & Well (1981) who used think-aloud protocols and follow-up interviews to examine computational versus conceptual understanding of the arithmetic mean for seventeen undergraduate students; Mokros and Russell (1995) engaged twenty-one subjects in task-based interviews to gather data about concepts of average; and Groth (2005) who used tasked-based clinical interviews to investigate the "intricate thinking processes" of fifteen subjects as they solved arithmetic mean problems. Examples of studies in mathematics education and statistics education that have utilized the methodology of verbal protocols to elicit data, but are not specifically about the arithmetic mean, are summarized in Table 3-1.

Table 3-1: Verbal Protocol Study Sample Sizes

Author(s)	Description	Sample Size
Clement, J., (1982)	Conducted think-aloud protocols of college students solving algebra word problems to find the cognitive processes connected with particular misconceptions.	15
Allwood, C.M., (1990)	Used think-aloud problem solving sessions in two studies to investigate the relationship between the justification of a choice of solution method and the correctness of that choice in statistical problems.	16 and 20
Montague, M. & Applegate, B., (1993)	Analyzed think-aloud protocols of middle-school students solving mathematical problems to identify solution methods of students with different abilities.	30
Thelk, A. & Hoole, E. (2006)	Investigated the cognitive validity of scientific and quantitative reasoning items using think-alouds collected from first-year university students.	27

The sample size for the current study was similar to the sample sizes of the eight studies described in Table 3-1. The verbal protocols provided rich detail regarding the knowledge used by the participants while solving the arithmetic mean problems. Although, the sample size was not sufficient to achieve the typically accepted power for quantitative statistical analysis, it was chosen to balance the quantitative and qualitative aspects of the study. In particular, the richness of the data provided by the verbal protocols compensated for the statistical deficiency of the sample size in the overall analysis.

3.3 DATA COLLECTION

This section details the procedures that were used in the collection of data. The methodology of *protocol analysis* (Newell & Simon, 1972) was employed to elicit knowledge of the particular conceptualization and domain accessed by participants while solving the arithmetic mean problems on the instrument described in section 3.3.1. Protocol analysis uses think-aloud sessions as a means for gathering data.

Protocol analysis is a rigorous methodology for eliciting verbal reports of thought sequences as a valid source of data on thinking (Ericsson, 2002). The idea of protocol analysis was first developed by Newell and Simon (1972). Ericsson and Simon (1980) first offered substantial empirical proof of the validity of verbal think-aloud protocols as data. That is, that verbalizing ones thoughts, without ancillary descriptions or explanations, does not alter the cognitive sequence of thought, but engaging in specific thought activities (i.e. describing or explaining) changes the cognitive process (Ericsson & Simon, 1993). Therefore, a key component of protocol analysis is the subject's ability to continually think aloud while participating in a task (e.g. problem solving) focusing solely on their solution process without being interrupted and asked to describe or explain their thoughts. Ericsson and Simon outline a systematic method for collecting the verbal data that is reliable and valid. This includes providing adequate instruction and practice for the participants, minimizing distractions in the research environment, and developing a clearly focused task.

For this study, two types of data were collected: (a) verbal protocols, along with (b) written artifacts to capture the attributes of the mathematical and statistical knowledge demonstrated during the problem solving process. The nonverbal, or written solution, documented the symbolic and pictorial thoughts of the participant and provided a familiar vehicle to stimulate the problem solving process. Schoenfeld (1985) points out that the advantage of a written artifact along with the verbal protocol in mathematical problem solving is that the verbalization alone "rarely serves to elucidate their [participants'] workings" (p. 282). The purpose of participant-generated verbal conceptualization during the problem solving activity was to gather the most accurate representation of the participant's thought process. An important feature of protocol analysis is that, unlike a clinical interview, it is a noninterventionist method used to eliminate the risk of altering the student's solution path and eliminate any potential for a researcher-participant generated training effect (Schoenfeld, 1985).

The noninterventionist data collection method of protocol analysis served as a controlled means to elicit the thought process of the participants. It provided an unobtrusive and research-grounded method to gather data that revealed the accessed knowledge and cognitive processes of the participants as they solved the arithmetic mean problems.

3.3.1 Instrument

Of principle interest in this research study was the cognitive relationship between the conceptualizations of fair-share and center-of-balance, as well as the cognitive relationship between these conceptualizations and the mathematical domain in a conceptually rich problem solving environment. Thus, the tasks or problems were designed to give participants the opportunity to demonstrate knowledge relevant to the conceptualization or domain of interest during the solution process.

Several sources were explored in an effort to locate problems that met the above criteria. These sources included:

1) Problems from previous research studies including:
   a) Cai, Moyer, & Grochowski, 1999
   b) MacCullough, 2007
   c) Mevarech, 1983
   d) Mokros & Russell, 1995
   e) Strauss & Bichler, 1988
2) Problems from published textbooks including:
   a) Aufmann, Lockwood, Nation, & Clegg, 2007
   b) Freedman, Pisani, & Purves, 1998
3) Problems generated by the author for particular use in this study
4) Problems proposed by an expert in statistical education for particular use in this study
5) Problems proposed by an expert in mathematics for particular use in this study

The problems were informally piloted and responses analyzed by the author and an independent expert in mathematics education. The problems were categorized into three groups based on the analysis of the detailed written solutions: (a) those connected to the conceptualization of fair-share, (b) those connected to the conceptualization of center-of-balance, and (c) those connected to mathematical concepts of the arithmetic mean. The three problems in each of the three categories, fair-share, center-of-balance, and mathematical concepts of the arithmetic mean, identified as most useful in revealing conceptual understanding were selected for use in a formal pilot study. One purpose of the pilot study was to select the best two problems within each category.

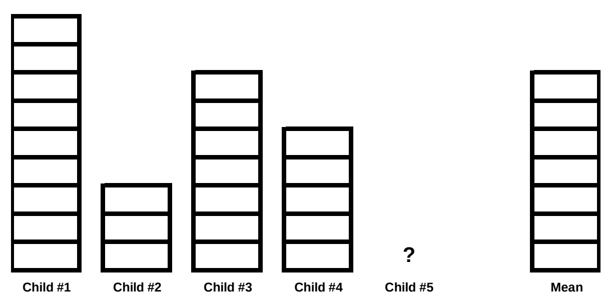
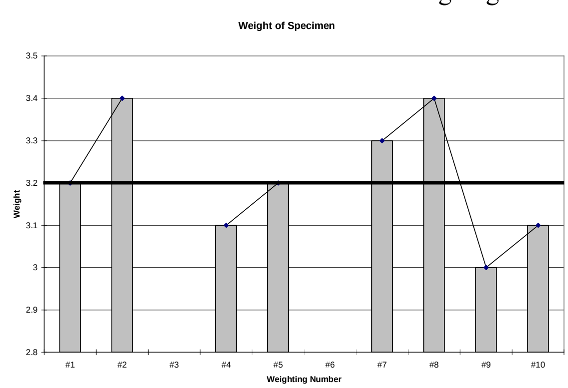
The pilot study adhered to the procedures for data collection, data coding, and data analysis that were used in the current study. The solutions (i.e. verbal protocols and written work) from the pilot study were analyzed on two dimensions:

1) Validity – The problem's solution elicited verbal protocols that contained identifiable knowledge segments.
2) Reliability – The problem's predominant solution approaches included knowledge relevant to the problem classification (i.e. fair-share, center-of-balance, or mathematical concepts)

On the basis of the pilot, two of the three problems in each category, for a total of six, were selected for use in the current study. These six problems are described in the subsequent sections while Appendix A identifies all of the problems piloted that were not used in the study and a rationale for their exclusion.

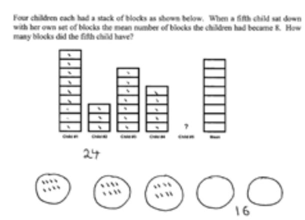
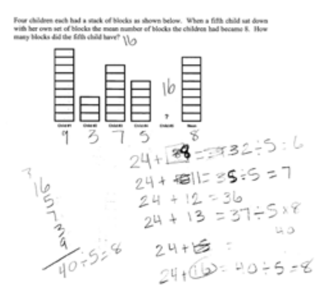
3.3.1.1 Fair-Share Problems

Fair-share problems are those tasks that were most likely to be solved using one of two models of fair-share, redistribution or signal-in-a-noisy-process. Many fair-share problems can be solved efficiently using the arithmetic mean formula. Such solutions could be the result of conceptual understanding of fair-share, or it is also plausible that such solutions are indicative of simply a procedural understanding of the arithmetic mean formula. Problems that involve constructing data from a given mean, rather than solving for a mean from given data, are more statistically challenging and are more likely to elicit a conceptual solution because they encumber the ability to simply use the arithmetic mean algorithm (Mokros & Russell, 1995; pilot). Therefore, the fair-share problems selected for this study were missing data problems associated with the models of redistribution and signal-in-a-noisy-process. The two problems used in this study are shown below.

Fair-Share Problem 1 **FS1**	Four children each had a stack of blocks as shown below. When a fifth child sat down with her own set of blocks the mean number of blocks the children had became seven. How many blocks did the fifth child have? 
Fair-Share Problem 2 **FS2**	In a chemistry lab a student weighed a specimen ten times. The results of those weighings are presented in the chart below. The student lost the 3rd and 6th weighings of the specimen after she calculated the mean of the ten weighings to be 3.2 as indicated by the dark line in the graph below. What could have been the values for the 3rd and 6th weighings if the mean is 3.2? 

FS1 is a task used by Cai, Moyer, and Grochowski (1999) in their research study on the conceptual understanding of average. Results from that study indicated students either used a block-leveling or redistribution approach based on a statistical understanding of the mean, an algebraic approach based on the arithmetic mean formula, or a guess-and-check strategy that was either conceptually or procedurally based. The different approaches varied with the type of instruction the students had previously experienced. Although the pilot work for this current study did not focus on prior instruction, participants' written solutions verified the three systematic approaches to solving the problem. The problem was piloted nine times. The results, presented in Table 3-2, show that the majority of solutions utilized, to varying degrees, knowledge of fair-share.

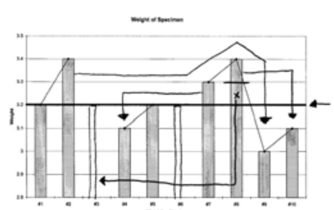
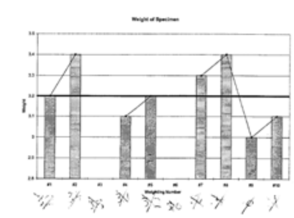
Table 3-2: FS1 Pilot Solutions

Method of Solution	Number of Occurrences	Description	Example
Fair-Share	7	Utilized some method of block leveling or redistributing the data so that each *n* had an equal amount; more precisely equal to the mean	
Mathematical	2	Utilized an algebraic approach to the arithmetic mean formula or a guess-and-check plan to find a solution	

The results obtained by Cai et al. (1999) and this pilot indicate problem FS1 is a valid task for obtaining data regarding knowledge utilized in solving the problem.

FS2 is based on a signal-in-a-noisy-process task originally designed by Konold and Pollatsek (2002) and used in a study by Groth (2005). Originally, only the line plot of all of the data points was given and students were asked to find the mean. This original version of the problem was piloted to ten students; seven of whom numerically translated the data into the arithmetic mean formula inhibiting the analysis of conceptual understanding. Bar lines were added and the students were asked to find missing data points to reduce the likelihood that it would be solved using a procedural computation. The current format was piloted to eight students; only one student attempted to use a formulaic procedure. The solution methods are indicated below in Table 3-3:

Table 3-3: FS2 Pilot Solutions

Method of Solution	Number of Occurrences	Description	Example
Fair-Share	6	Utilized some method of redistributing the data so that each *n* had an equal amount; more precisely equal to the mean	
Center-of-Balance	1	Utilized a method that indicate the mean was a "balancing point" with equal amount of data above and below the mean	
Mathematical	1	Utilized arithmetic mean formula to solve for missing data points	$\frac{3.2+3.4+\#3+3.1+3.2+\#6+3.3+3.4+3+3.1}{9}=3.2$ $\frac{25.7+\#3+\#6}{9}=3.2$ $2.8+\#3+\#6=3.2$ $\#3=.2$ $\#6=.2$

The results obtained by Konold and Pollatsek (2002) and Groth (2005) indicate signal-in-a-noisy-process problems, similar to FS2, are valid indicators of fair-share knowledge in arithmetic mean problems.

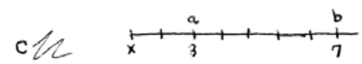
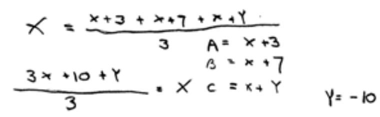
3.3.1.2 Center-of Balance Problems

Center-of-balance problems are characterized by solution methods involving the idea of balance or use of a block-stacking model. As with fair-share problems, many center-of-balance problems can be solved effectively using the arithmetic mean formula. Again, such solutions could indicate a conceptual understanding of center-of-balance or simply a rote understanding of the arithmetic mean formula. Consequently, center-of-balance problems were selected after problems in previous research studies and problems from piloted instruments were carefully analyzed.

Center of Balance Problem 1 **CB1**	Given three numbers, (a,b,c), and the mean of these numbers is $\bar{x}$. We know that a is 3 greater than $\bar{x}$ and b is 7 greater than $\bar{x}$. How does the value of c relate to $\bar{x}$?
Center of Balance Problem 2 **CB2**	As a worker in a grocery store you are asked to place price stickers on nine bags of potato chips so that the mean price of the chips is \$1.38. You can not price any bag at exactly \$1.38. You also must price one bag at \$1.30 and a second bag at \$1.35. Create the remaining seven price stickers

CB1 was developed by the researcher for use in this study. It was piloted twelve times with written solutions in two separate problem solving sessions and three times using verbal protocols. Table 3-4 details the results of the pilot work.

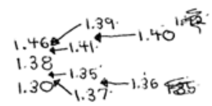
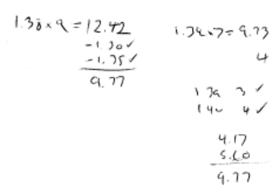
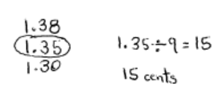
Table 3-4: CB1 Pilot Solutions

Method of Solution	Number of Occurrences	Description	Example
Center-of-Balance	12	Used some method of balancing or centering the data around the mean.	
Mathematical	3	Used algebraic equation to correctly solve the problem (1). Used nonsensical arithmetic to incorrectly attempt the problem (2), not shown.	

The consistency of the results in all three pilot applications suggests it is a valid task for measuring the conceptualization of center-of-balance knowledge of the arithmetic mean.

CB2 is an amalgamation of two problems used by Mokros and Russell (1995) to study students' concepts of average and representativeness. One problem in their study made use of the potato chip pricing context describe in CB2, while a second problem retained the pricing constraints of CB2. They found solutions could be categorized into two groups: (a) those that indicated a non-representative nature of the arithmetic mean, and (b) those that indicated the arithmetic mean as a representative number. The non-representative nature of the arithmetic mean was most often manifest as understanding the mean to only be an algorithmic process. Those who saw the mean as a representative number indicated the mean as a center-of-balance or a reasonable mathematical representation of the data set. The problem was successfully used by Mokros and Russell to determine the knowledge used in the solution process and, therefore, is a valid measure of such knowledge. The problem was piloted three times using verbal protocols and three times with detailed written solutions as shown in Table 3-5.

Table 3-5: CB2 Pilot Solutions

Method of Solution	Number of Occurrences	Description	Example
Center-of-Balance	4	Used some method of balancing or centering the data around the mean.	
Fair-Share	1	Used total sum of data points and a redistribution to correctly solve the problem.	
Mathematical	1	Used nonsensical arithmetic to incorrectly attempt the problem.	

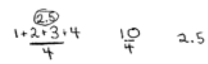
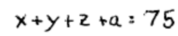
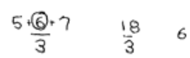
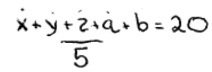
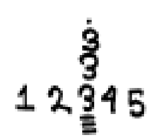
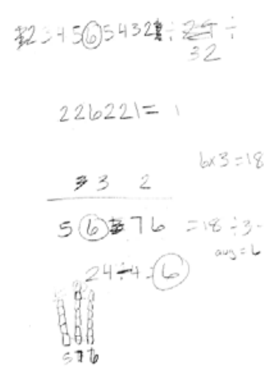
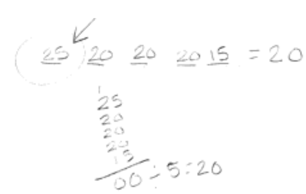
3.3.1.3 Mathematical Concept Problems

Mathematical concept problems were developed to elicit knowledge of particular mathematical concepts integral to the arithmetic mean. The mathematical concept problems attempt to capture participants' use of mathematical concepts while solving problems about the arithmetic mean.

Mathematical Concept Problem 1 **MC1**	What value can be added to a data set so that the arithmetic mean of the data set does not change? Why?
Mathematical Concept Problem 2 **MC2**	The mean of five numbers is 20. The sum of four of these numbers is 75. What is the value of the fifth number?

MC1 was piloted on two informal pilot instruments and in the pilot study. Results from all three pilots indicated participants were most successful when their mathematical solutions were guided by the statistical conceptualization of center-of-balance. Results from the pilot study for MC2 indicated participants were more successful in conceptually developing a solution if they couched the mathematics within the statistical conceptualizations of fair-share. Table 3-6 summarizes the results of the pilot work for the mathematical concept problems.

Table 3-6: MC Pilot Solutions

Method of Solution	Description	Examples: MC1	Examples: MC2
Mathematical	MC1: Stated the middle number was the answer and used inductive reasoning to show proof.		
	MC2: Unsuccessfully used the arithmetic mean formula to find the missing data point.		
Mathematical Fair-Share	MC2: Used notion of fair-share to calculate total sum was 100. Subtracted total sum of four numbers to find the fifth.	no example	
Mathematical Center of Balance	MC1: Used center-of-balance example to illustrate one could add the mean multiple times to a data set and not change the mean		no example
Mathematical Mixed FS and CB	MC1: Struggled to use an example based on center-of-balance to illustrate the mean was the answer. Used a second fair-share strategy to effectively demonstrate inductive proof. MC2: Used notion of fair-share to create a data set. Used notion of center-of-balance to center data round the mean.		

3.3.1.4 Assignment of Problems

Problem order was assigned to the pretest and posttest by purposeful random selection. The first problem participants solved was randomly selected from the two problems of the same statistical conceptualization (i.e. fair-share or center-of-balance) as their respective instruction. A problem based on the conceptualization respective to instruction was chosen first to precipitate use of that knowledge in the problem solving process. The next three problems alternated between the two statistical conceptualizations to minimize direct carry-over solution strategies. The last two problems on the instruments were randomly ordered mathematical concept problems. The mathematical concept problems were offered last to facilitate the possibility of multiple conceptualizations within their solutions. The identical pretest and posttest were administered to all participants in the fair-share and center-of-balance instructional groups. The control group randomly received one of the two, fair-share instructional group's or center-of-balance instructional group's, versions of the instrument. Table 3-7 summarizes the order of problems based on instructional groups.

Table 3-7: Assignment of Problems

Fair-Share Group	Center-of-Balance Group
FS1	CB1
CB1	FS2
FS2	CB2
CB2	FS1
MC2	MC2
MC1	MC1

These six problems were selected because their constructs compelled participants to utilize knowledge from a particular knowledge domain, mathematical or statistical, or from a particular conceptualization, fair-share or center-of-balance. The pilot work revealed various correct approaches to solving the problems along with many incorrect paths or ill-conceived starting points. It also suggested that participants who understood the representative nature of the arithmetic mean were able to use the conceptualizations of fair-share and center-of-balance or mathematical concepts related to these conceptualizations to create solutions with varying representations.

3.3.2 Procedure

In this study each participant completed a pretest consisting of a think-aloud problem solving session of the tasks on the instrument described in section 3.3.1. Next, approximately two weeks later, each participant was randomly assigned to one of three instructional treatment groups, fair-share, center-of-balance, or control. The fair-share group received individual instruction associated with fair-share knowledge and related mathematical concepts integral to the arithmetic mean. Similarly, the second instructional group, center-of balance, received individual instruction associated with center-of-balance knowledge and related mathematical concepts as they pertain to the arithmetic mean. The control group received individual instruction of general problem solving heuristics. Last, within one week of exposure to the instruction, each participant completed a posttest think-aloud problem solving session identical to the pretest. Twenty-nine of the thirty initially chosen participants completed all three phases of the study: pretest, instruction, and posttest. Figure 3-1 depicts the general model of the research design and procedure.

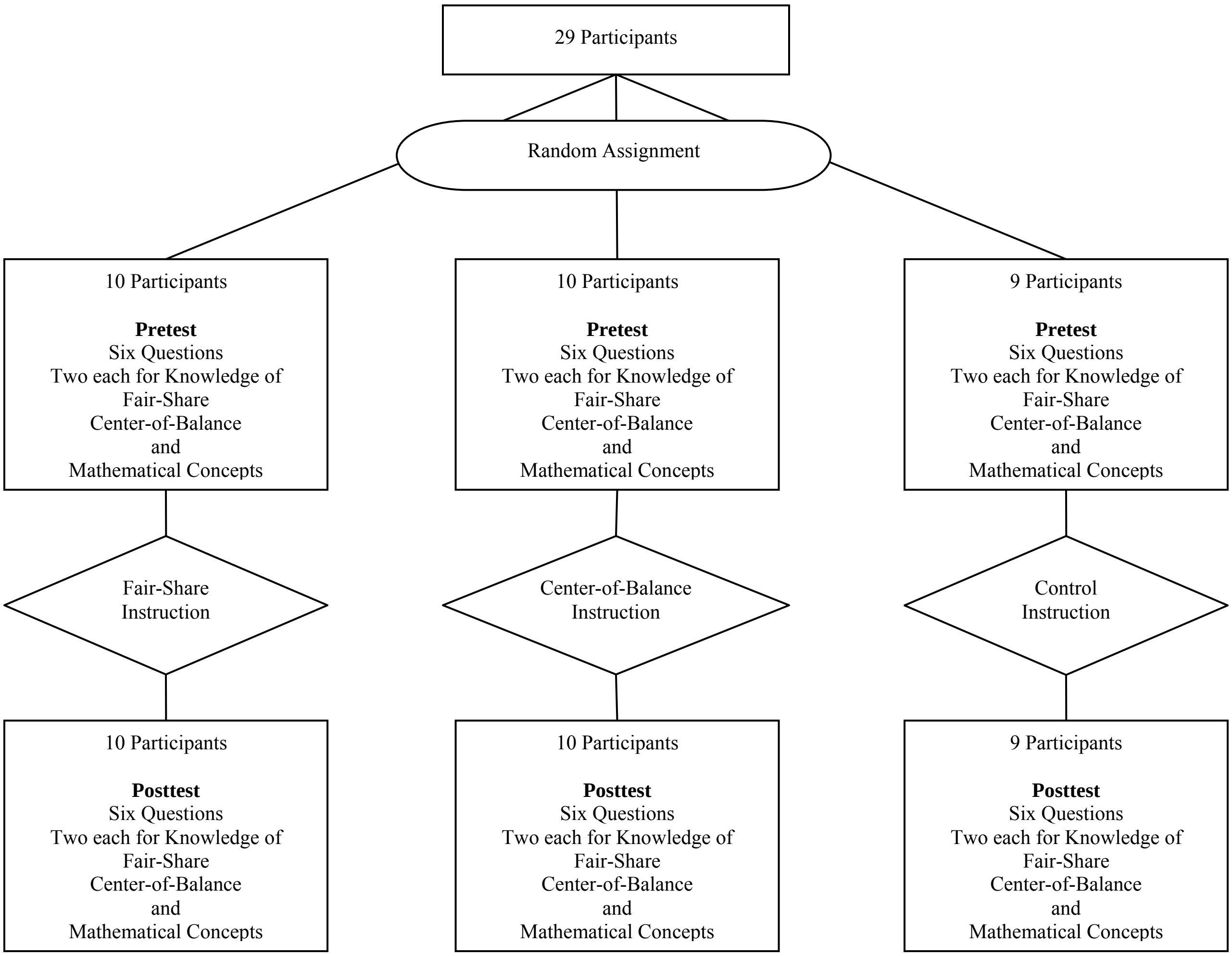


Figure 3-1: Research Design

3.3.2.1 Verbal Protocol Pretest and Posttest

Participants completed a short training session to become familiar with the necessary elements of the study and think-aloud protocols. The training session included an introduction to the study, explanation of the consent form, and instructions regarding think-aloud problem solving. The specific instructions consisted of a prepared account of how to think aloud without explaining thoughts (see Appendix B.1) (Ericsson & Simon, 1993). The participants practiced with three carefully chosen warm-up exercises to acclimate themselves to the verbal protocol process without producing a practice effect (see Appendix B.2) (Ericsson & Simon, 1993). During the warm-up exercises the participants were advised if they were not verbalizing and/or if their verbalizations were not relating to their thought process and were asked to make appropriate adjustments to their verbal reporting. Ericsson and Simon (1993) found, "it is very rare that subjects do not spontaneously verbalize in a normal fashion after a couple of practice problems" (p. 377) and "[warm-up tasks]…eliminate silence due to misunderstanding of instruction to think aloud" and "give subjects practice in expressing thoughts directly without explaining or interrelating the information" (p. 257).

The pretest problems were individually administered to students immediately after they completed the training session. Each student met in an empty conference room and was seated in front of and not facing the examiner (Ericsson & Simon, 1993). The students were reread the instructions from the training session on proper think-aloud procedures and were asked if they had any further questions. The sessions were audio recorded for analysis and transcription. The problem solving activities were not timed and students were encouraged to work until they reached a solution or could make no further progress on a particular problem. The only communication with the examiner during actual problem solving was a reminder to "keep talking" if the student was silent for more than five seconds (Ericsson & Simon, 1993; Montague & Applegate, 1993). The procedures for the posttest were the same as those of the pretest sessions.

3.3.2.2 Knowledge Instruction Modules

Three instruction modules (IMs), (a) knowledge of fair-share with related mathematical concepts, (b) knowledge of center-of-balance with related mathematical concepts, and (c) a general problem solving module were created for use in this study. The purpose of the IMs was to increase the knowledge of a particular conceptualization of the arithmetic mean (i.e. fair-share or center-of-balance) and to relate that conceptualization to mathematical concepts, such as 'the sum of the deviations from the mean is zero.' The modules were accessed and completed by the participants using the Blackboard online learning system. Each participant was given access to one module and progress toward completion of the module was tracked by the researcher using the capabilities of the Blackboard system. Each IM utilized several teaching strategies and multiple modes of content presentation to broaden its capacity to reach diversified learning styles and ability levels. Appendix C offers the general composition of the knowledge instruction modules.

The fair-share module was developed to introduce the arithmetic mean as an equal allocation of data and relate it to the mathematical property proposed by Strauss and Bichler (1988), the sum of the deviations of the data from the mean is zero. The module utilized notes and examples that accompanied interactive exercises and video segments of instruction of the fair-share conceptualization.

The center-of-balance module emphasized the arithmetic mean as the balancing point of a data set. The module was similar to the fair-share module in that it utilized interactive instruction, multiple exercises, and video clips of the arithmetic mean being taught as a center-of-balance. The module related the center-of-balance conceptualization to equalizing data above and below the mean (i.e. a model depicting the sum of the deviation of the data from the mean is zero).

The control module focused on general mathematical problem solving skills. The content was based on Polya's (1957) *How to Solve It* four steps of problem solving: understand the problem, devise a plan, carry out the plan, and look back and check answer. The module covered different problem solving strategies, such as inductive reasoning, trial-and-error, illustration, and related problems. The learning material for the module was taken from the liberal arts mathematical course text.

The instruction modules were designed to simulate the level and amount of instruction typically devoted to the arithmetic mean in a liberal arts mathematics course. The students in a liberal arts mathematics course have varying degrees of prior knowledge in mathematics, and, in particular, knowledge concerning the arithmetic mean. Therefore, discrete data sets are often used as examples to conceptually develop the notion of the arithmetic mean. Typically, one lecture, or one-and-a-half hours, is reserved for teaching statistical measures of center (i.e. mean, median, and mode). Most often, homework problems are assigned and discussed in a subsequent class. The presentation of the notes and the video segments of the IMs corresponded to a conventional lecture in a traditional classroom; the examples in the IMs were typical of class examples, and the interactive exercises characterized homework problems and feedback. The level of instruction, discrete examples that are accessible by varying skill levels, and the amount of time, a portion of one lecture, aligned the instruction modules with typical instruction regarding the arithmetic mean in a liberal arts mathematics course.

3.3.3 Data Collection Summary

The participants for this study were university students enrolled in a liberal arts mathematics course. Twenty-nine participants completed a three phase data collection process: pretest, instruction, and posttest. Pre-session training instructed participants on the think-aloud problem solving method. The data collected from the pretest included verbal protocols along with written solutions to the six arithmetic mean problems. The participants were randomly divided into three groups; each receiving an instructional module in either fair-share or center-of-balance, or participating in a control group. Following the instruction, participants completed a posttest with identical problems to those on the pretest to preserve validity and reliability. A control group was utilized to manage any perceived learning effect. The collected verbal protocols of the pretest and posttest were fully transcribed and organized in preparation for data coding.

3.4 DATA CODING

Chi (1997) offered a practical guide to quantifying verbal data. She described four functional steps that can be applied to transforming evidence of knowledge in the verbal protocols into appropriate numerical scores based on a specified rubric. These steps for coding the data are:

1) Searching and segmenting the protocols
2) Developing a coding scheme and formalism (rubric)
3) Operationalizing evidence in coded protocols to the formalism
4) Depicting or summarizing the data

This section will delineate the methods used in the current study to achieve the proposed list of functional steps.

The transcribed protocols were *searched* for occurrences of mathematical and statistical knowledge and *segmented* between shifts in knowledge domains or conceptualizations. Each segment depicted the qualities of an individually represented concept or multiple concepts that occured simultaneously or were conceivably integrated within or across domains. The segment may have been either correct or based on false beliefs and represented a single mathematical or statistical concept or an integrated group of concepts, regardless of the length of verbalization that generated it. Each of the six problems for each participant was individually searched and segmented.

The *coding scheme* used in this study was developed to correspond to the proposed knowledge structure offered in section 2.4. Chi reports that knowledge data is best coded as the elements or nodes of a predetermined network. The codes depicted in Table 3-8 were developed to represent the various domains and concepts of knowledge related to the arithmetic mean.

Table 3-8: Coding Scheme

Domain	Conceptualization
S → statistical	
	FS → fair-share
	CB → center-of-balance
M → mathematical	

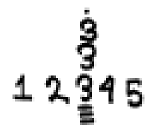
Each segment in the verbal protocols was coded with a domain and/or conceptualization symbol to demonstrate a participant accessed or utilized that knowledge in the problem solving process. Appendix D gives three examples of searched and coded protocols from the pilot study.

The coded protocols were utilized in both the quantitative and qualitative aspects of the study. They helped categorize the knowledge that was scored for quantitative analysis and identified segments of knowledge that were compared in the qualitative analysis.

Operationalizing evidence in the verbal protocols consisted of assigning an appropriate numeric score based on a descriptive rubric. A scoring rubric was developed for each class of problem (i.e fair-share, center-of-balance, and mathematical concepts). The rubrics scored participant responses on a discrete scale of zero to three (0-3). A "zero" score was given if a participant failed to attempt a solution or did not include any mathematically or statistically sound knowledge in their solution. A "three" score represented sound use of mathematical and/or statistical knowledge that signified a conceptually correct solution. The score was based only on the level of knowledge demonstrated of a particular type for a particular class of problem; that is, only fair-share knowledge was scored for fair-share problems, and only center-of-balance knowledge was scored for center-of-balance problems. If a participant used a viable method to solve a problem that did not correspond with the classification of the problem, a "no-score" was given to that problem and the viable method used to solve the problem was noted for further analysis. For each score, each rubric has a description of the knowledge as well as example solutions that correspond to that score.

Table 3-9 is the mathematical concept rubric. It was adapted from the general scoring rubric developed for assessing QCAI (QUASAR Cognitive Assessment Instrument) tasks (Lane, 1993). The general scoring rubric includes knowledge of mathematical concepts and the integration of those concepts into other models or elements. For the purposes of this study the integration of mathematical knowledge may have been with other mathematical concepts or with the conceptualizations of fair-share and center-of-balance.

Table 3-9: Mathematical Concepts Rubric

Score	Description	Example	Example
3	Shows understanding of the problem's mathematical concepts and principles; and executes algorithms completely or at worst with minor errors. May use relevant information of a formal or informal nature; identifies all the important elements of the problem and shows understanding of the relationships between them; reflects an appropriate and systematic strategy for solving the problem; and gives evidence of a solution process, and solution process is at worst nearly complete and systematic.	Used center-of-balance example to illustrate one could add the mean multiple times and to a data set and not change the mean 	Used notion of center-of-balance to center data around the mean. 25 20 20 20 15 = 20 25 20 20 20 15 100 ÷ 5 = 20
2	Shows understanding of some of the problem's mathematical concepts and principles; and may contain serious computational errors. Identifies some important elements of the problems but shows only limited understanding of the relationships between them; and gives some evidence of a solution process, but solution process may be incomplete or somewhat unsystematic.	Correctly used inductive examples to demonstrate point. Could not generalize work to all cases of the mean and states the answer is the "middle" number of a data set. $\frac{1+2+3+4}{4}$ $\frac{10}{4}$ 2.5 $\frac{5+6+7}{3}$ $\frac{18}{3}$ 6	Unsuccessfully used the arithmetic mean formula to find the missing data point, but illustrated correct algebraic notation and the relationship of the problem to the arithmetic mean formula. $x+y+z+a=75$ $\frac{x+y+z+a+b}{5}=20$
1	Shows very limited understanding of the problem's mathematical concepts and principles; may misuse or fail to use mathematical terms; and may make major computational errors. May attempt to use irrelevant outside information; fails to identify important elements or places too much emphasis on unimportant elements; may reflect an inappropriate strategy for solving the problem; solution process may be missing, difficult to identify, or completely unsystematic.	Used inductive example to illustrate the mean of a group of numbers, added zero to the data set and used the same calculation to erroneously calculate the same mean. 1 2 2 0 $\frac{6}{3}=2$	Illustrated relevance of arithmetic mean formula to the problem but showed no understanding of how to use it to solve the problem. $\frac{\text{add num}}{5}=20$
0	Shows no understanding of the problem's mathematical concepts and principles. May attempt to use irrelevant outside information; fails to indicate which elements of the problem are appropriate; copies part of the problem, but without attempting a solution.	Uses the arithmetic mean formula but makes no connection to the problem or progress toward a solution What value can be added to a data set so that the arithmetic mean of the data set does not change? $5+6+3=\frac{14}{3}=\boxed{4.6}$	Writes down the arithmetic mean formula but not in terms of values in the problem. $\frac{\text{the sum of the data}}{\text{number of items}}$ ÷ = mean

(adapted from Lane, 1993)

Table 3-10 and Table 3-11 show the fair-share and center-of-balance rubrics, respectively. They were developed as specializations of the general scoring rubric (Lane, 1993) based on analysis of pilot work and pilot study problems. The descriptions of knowledge were devised based on the range of knowledge demonstrated in the pilot problems. The score indicates not only the correctness, but the understanding of the respective statistical conceptualization. That is, not only was there a viable solution, but that solution was found using sound mathematical and statistical concepts. Therefore, a rubric score was based on two elements: (a) whether a particular statistical conceptualization of the arithmetic mean was used in the solution and (b) the soundness of the mathematics and statistics utilized in the solution.

Table 3-10: Fair-Share Rubric

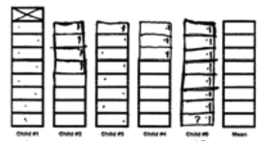
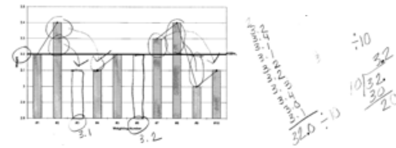
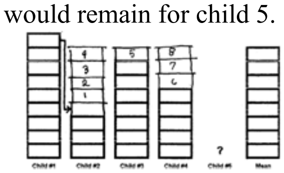
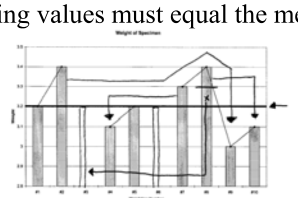
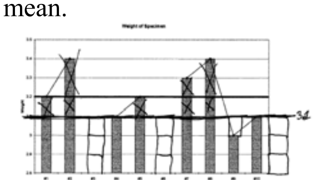
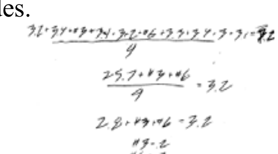
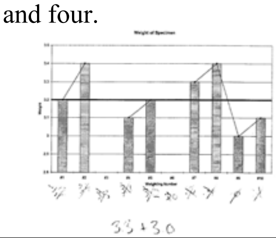
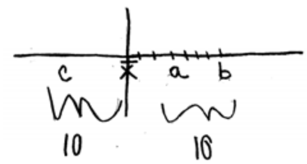
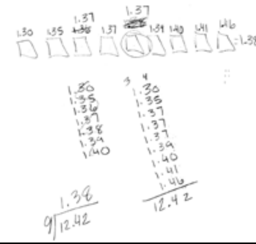
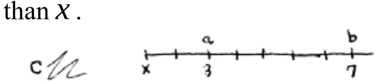
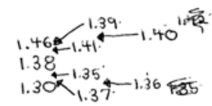
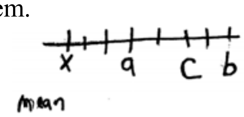
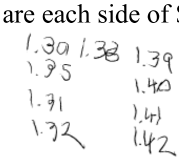
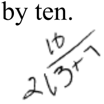
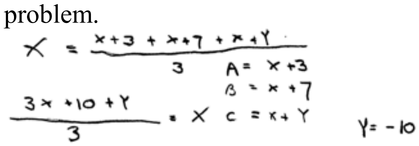
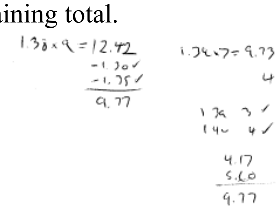
Score	Description	Example	Example
3	Makes use of knowledge of fair-share in a statistically and mathematically sound approach to a complete solution with or without arithmetic errors	Removed blocks from stack with more than the mean. Adds blocks to stacks less than the mean to equal the mean. Places the mean number of blocks in child 5 stack. Counts added blocks 	Redistributed the values above the mean line to the values below the line so that each bar was the same height. Checks work. 
2	Makes some use of knowledge of fair-share in a statistically and mathematically sound approach toward a solution but deviates from conceptual soundness or is unable to complete a solution.	Removed blocks from stack with more than the mean. Adds blocks to stacks less than the mean to equal the mean. Counts added blocks failing to add the mean number of blocks that would remain for child 5. 	Redistributed the values above the mean line to the values below the line so that each bar is the same height. Failed to account for the value distributed to the number three weighing and answers the two missing values must equal the mean. 
1	Indicates knowledge of the concept of fair-share but does not apply it or grossly misapplies it to a solution.	Used blocks in the stack representing the mean along with blocks in the other stacks to find an estimated new mean. 	Made all the bars representing data points the same height but unequal to the mean. 
0	Does not indicate any mathematically or statistically sound knowledge in the solution.	Added the number of blocks of the first four children and the mean, 9+3+7+5+8=32. Divides the result by 5 to get answer. (Cai et al., 1999)	Used formula to add each value and the missing two values. Incorrectly divides. 
_	Makes use of a viable method to solve the problem that does not include knowledge of fair-share.	Used algebraic equation to solve the problem after counting blocks to get data values. $9+3+7+5+x=8$ $\frac{24+x}{5}=8$ $24+x=40$ $x=16$	Balanced each value above the mean with a value below the mean. Answers balance weighing numbers two and four. 

Table 3-11: Center-of-Balance Rubric

Score	Description	Example	Example
3	Makes use of knowledge of center-of-balance in a statistically and mathematically sound approach to a complete solution with or without arithmetic errors	Placed *a* three units and *b* seven units from $\bar{x}$ on a balance drawing. Indicated *c* must equal the combined distance *a* and *b* are away from $\bar{x}$ but on the other side of the fulcrum. 	Drew blocks to represent bags. Assigned given values of \$1.30 and \$1.35 with \$1.38 (the mean) as the middle. Filled in values for remaining blocks that balance on each side of \$1.38. Made a correction to the \$1.38 block since that value can not be used. Checked work. 
2	Makes some use of knowledge of center-of-balance in a statistically and mathematically sound approach toward a solution but deviates from conceptual soundness or is unable to complete a solution.	Drew a number line and correctly placed *a*, *b*, and $\bar{x}$ in their relative places. Illustrated *c* must be on the opposite side of *a* and *b* but only referred to *c* as having a value less than $\bar{x}$. 	Recognized \$1.38 is the mean. Compensated for the given values of \$1.30 and \$1.35 on the opposite side of the mean. Continued process with other values, but was unable to adjust for the constraint of not using \$1.38. 
1	Indicates knowledge of the concept of center-of-balance but does not apply it or grossly misapplies it to a solution.	Drew a number line with *a* and *b* three and seven units respectively from $\bar{x}$. Incorrectly placed *c* on the number line between *a* and *b* believing this "balanced" the problem. 	Stated \$1.38 needs to be the middle number. Randomly picks numbers higher and lower than \$1.38 so that there are an equal number of numbers are each side of \$1.38. 
0	Does not indicate any mathematically or statistically sound knowledge in the solution.	Added three and seven together. Divided sum by ten. 	Found median of given numbers. Picked \$1.35 and divides by nine. Presented answer as \$.15 per bag. (Mokros & Russell, 1995)
–	Makes use of a viable method to solve the problem that does not include knowledge of center-of-balance	Used algebraic equation to solve the problem. 	Found the total cost of nine bags of chips. Subtracted \$1.30 and 1.35. Found seven bags that add to the remaining total. 

Participants' written work and verbal protocol for each problem were examined holistically for use of a particular knowledge and for level of understanding of that knowledge. This was then compared with the descriptions of knowledge that corresponded to scoring levels delineated in the respective rubrics.

A scoring bundle for each participant was created to efficiently *depict and summarize the data*. The bundle included a scoring sheet for each of the participant's protocols (see Appendix E.1). The scoring sheet included a side-by-side comparison of the pre- and post- test coded verbal protocols, the rubric score for each problem, and comments by the researcher and coder for use in qualitative analysis. Each participant was assigned three scores for the pretest problems corresponding to the three knowledge types, fair-share, center-of-balance, and mathematical concepts. The scores were calculated by finding the mean score of the two problems of each knowledge type on the pretest. The same process was used to find three scores corresponding to the same knowledge types on the posttest. Figure 3-2 shows how the quantitative scores for each participant were summarized.

<table>
<tr><th colspan="9">Participant #</th></tr>
<tr><th colspan="4">Pretest</th><th rowspan="2">Gain Score</th><th colspan="4">Posttest</th></tr>
<tr><th>Problem Knowledge</th><th>Problem</th><th>Score</th><th>Average</th><th>Average</th><th>Score</th><th>Problem</th><th>Problem Knowledge</th></tr>
<tr><td rowspan="2">Fair Share</td><td>FS1</td><td></td><td rowspan="2"></td><td rowspan="2"></td><td rowspan="2"></td><td></td><td>FS1</td><td rowspan="2">Fair Share</td></tr>
<tr><td>FS2</td><td></td><td></td><td>FS2</td></tr>
<tr><td rowspan="2">Center of Balance</td><td>CB1</td><td></td><td rowspan="2"></td><td rowspan="2"></td><td rowspan="2"></td><td></td><td>CB1</td><td rowspan="2">Center of Balance</td></tr>
<tr><td>CB2</td><td></td><td></td><td>CB2</td></tr>
<tr><td rowspan="2">Mathematical Concepts</td><td>MC1</td><td></td><td rowspan="2"></td><td rowspan="2"></td><td rowspan="2"></td><td></td><td>MC1</td><td rowspan="2">Mathematical Concepts</td></tr>
<tr><td>MC2</td><td></td><td></td><td>MC2</td></tr>
</table>

Figure 3-2: Participant Scoring Summary

The depiction and summary of the scores in these formats logically arranged the data for statistical analysis.

The procedure for coding and scoring the knowledge demonstrated in the verbal protocols adhered to guidelines Chi (1997) prescribed. Previous research studies that utilized these procedures (Chi & VanLehn, 1991; Griel, 1996; Montague & Applegate, 1993) indicated the method was a reliable and valid process for coding verbal protocols. Furthermore, inter-rater agreement was calculated for both the coding and scoring to verify reliability of the coded and scored protocols. The next section details the results of the inter-rater reliability comparisons.

3.4.1 Inter-rater Reliability

The participants' protocols (i.e. transcripts and written work) were coded anonymously with respect to pretest, posttest, and instructional group. Two coders, the author and an experienced educator with a graduate degree in mathematics and science education specializing in assessment, independently coded the pilot study protocols. Prior to coding, the coder was instructed in the coding process. This training included reading chapters 1-3 of this document, discussing the proposed knowledge structure for the arithmetic mean with the author, reviewing examples of protocols from the pilot study, and understanding the coding scheme and scoring rubrics. The author and coder first scored fifteen protocols from the pilot study. Initial inter-rater reliability was 62% for coding the predominant type of knowledge each protocol utilized and 73.3% (11/15) for scoring the protocols. Subsequent discussion of the proposed knowledge structure for the arithmetic mean's domains and a reevaluation of the protocols enabled the coders to agree on 88% of the coded segments and 93.3% (14/15) of the rubric scores. Another fifteen protocols of the pilot study were then independently coded and scored. The second examination resulted in reliabilities for 70% of knowledge used in each protocol and 80.0% (12/15) for rubric scores. Further discussion of the rubric scores resulted in 100% agreement between the two coders. A third set of eight protocols were coded, and reliabilities of 87.5% for coded segments and 87.5% (7/8) for rubric scores were achieved. See Appendix E.2 for completed scoring sheets from the pilot study that correspond to several of the examples displayed in the rubric Table 3-9, Table 3-10, and Table 3-11. The highlighted areas of the protocols signify key phrases or words that were used to differentiate the conceptualizations of fair-share and center-of-balance.

3.5 DATA ANALYSIS

This section describes the ways in which the data from the coded protocols was analyzed. First, the scores from pretest and posttest data were used to quantitatively examine any statistically significant connection between the conceptualizations of fair-share and center-of-balance, and any connection between mathematical concepts and the two conceptualizations of the arithmetic mean. Second, a qualitative analysis of the verbal protocols was conducted to describe any cognitive connection between fair-share and center-of-balance, and between these conceptualizations and the mathematical domain.

3.5.1 Quantitative Statistical Analysis

Two different statistical tools were used to quantitatively analyze the data. *Contingency tables* and the statistical model of *analysis of covariance* were employed to describe and analyze various aspects of the coded data. Examining the results of each statistical tool, independently and jointly, provided insight into the interactions and connections of the conceptualizations and domains proposed in the knowledge structure for the arithmetic mean. The two statistical tools, which examined the data from two perspectives, within each knowledge instruction group and between the knowledge instruction groups, provided adequate insight to answer the proposed research questions.

3.5.1.1 Association Study within Groups

A contingency table is a two-dimensional table (in the case of this study) in which each observation is classified on the basis of two variables simultaneously (Howell, 2002). Contingency tables are often used to simplify data by converting quantitative variables to categorical ones (N. Pfenning, personal communication, June 10, 2008).

This study examined whether instruction of one conceptualization of the arithmetic mean (i.e. fair-share and center-of-balance) affected knowledge of the other conceptualization. The participants were categorized with a “Yes” if they increased their score from the pretest to the posttest for a particular problem group that measured a specific knowledge (i.e. fair-share or center-of-balance), or a “No” if they did not. Within each instructional group (i.e. fair-share, center-of-balance) the data was arranged to denote gains in knowledge as demonstrated by increased problem scores. A 2x2 contingency table for relevant comparisons in each instruction group was reported and discussed.

Figure 3-3 depicts the contingency table for each instruction group. Each table compared several possible combinations within each group. The actual number of participants and corresponding percentage for each category were reported within each cell. The contingency tables indicated if a change in knowledge for one conceptualization, either fair-share or center-of-balance, of the arithmetic mean was related to a change in knowledge of the other conceptualization.

Fair-Share Instruction Group

		CB Knowledge	
	Increase	NO	YES
FS Knowledge	NO		
	YES		

Center-of-Balance Instruction Group

		FS Knowledge	
	Increase	NO	YES
CB Knowledge	NO		
	YES		

Figure 3-3: Contingency Tables

3.5.1.2 Comparison Study between Groups

Analysis of covariance (ANCOVA) is a statistical model developed by Sir Ronald Fisher in 1932 based on the precepts of the analysis of variance (ANOVA) model he created in 1925. Like the ANOVA, the ANCOVA offers a less likely chance of obtaining a type I error and incorrectly rejecting a true null hypothesis compared to performing multiple independent *t*-tests on more than two group means. The major difference between the two models is the addition of the covariate variable as a statistical control, thus combining ANOVA with regression analysis (Glass & Hopkins, 1996). “The covariate is defined as a source of variation not controlled for in the design of the experiment, but the researcher believes to affect the dependant variable. The covariate is used to statistically adjust the dependant variable” (Lomax, 2007, p. 84). The adjustment is made to the group means of the dependant variable, thus reducing statistical error. In the analysis of covariance, the group means of the covariant, as well as the linear relationship between the covariant and dependent variable are taken into account during the statistical analysis (Lomax, 2007). ANCOVA is an ANOVA on the statistically adjusted means. The price of using an ANCOVA compared to an ANOVA is the loss of one degree of freedom for each covariate; this results in more difficulty finding a significant test statistic. The appropriate use of analysis provides an economical method of comparing multiple group means. The analysis of covariance also allows for *multiple comparison* procedures for pairs of group means within the original larger group of means. Formulating the multiple comparisons within the presence of the ANCOVA helps to better control the power of the test and account for possible errors. In this study, the importance of the null hypothesis (group means for a particular type of knowledge are the same) for the omnibus *F*-test that examines all possible comparisons of the ANCOVA was simply a channel to perform the planned multiple comparison contrasts of more substantive interest.

In this study, the analysis of covariance compared posttest scores between the instruction groups. Differences between groups with instruction in knowledge of different statistical conceptualizations of the arithmetic mean were of particular interest in gaining insight about the cognitive relationship between the different conceptualizations and between the conceptualizations and mathematical concepts. The scores from the fair-share problems of the participants of the fair-share instruction group were compared to the fair-share problem scores of the participants in the center-of-balance instruction group and to the fair-share problem scores of the participants in the control group using an ANCOVA. A second and third ANCOVA were computed to perform a similar analysis on the scores from the center-of-balance problems and the scores from the mathematical concept problems. The following three hypotheses were tested where the posttest scores are denoted as FS for fair-share problems, CB for center-of-balance problems, and MC for mathematical concept problems; and the instructional groups are denoted as FSI for fair-share instruction, CBI for center-of-balance instruction, and CON for the control group instruction:

H_0: $FS_{FSI} = FS_{CBI} = FS_{CON}$	H_0: $CB_{FSI} = CB_{CBI} = CB_{CON}$	H_0: $MC_{FSI} = MC_{CBI} = MC_{CON}$
H_a: not all means are equal	H_a: not all means are equal	H_a: not all means are equal

For example, the first hypothesis tested the posttest scores of the fair-share problems across the three instructional groups, fair-share instruction, center-of-balance instruction, and control instruction. The second and third hypotheses followed a similar format but with center-of-balance and mathematical concept problems, respectively.

The pretest scores for each type of problem were used as the covariant for each analysis of covariance; that is, fair-share pretest problem scores were used as the covariant for the ANCOVA testing the hypothesis involving fair-share posttest scores, center-of-balance pretest problem scores were used as the covariant for the ANCOVA testing the hypothesis involving center-of-balance posttest scores, and mathematical concept pretest problem scores were used as the covariant for the ANCOVA testing the hypothesis involving mathematical concept posttest scores.

Following the three analyses of covariance, multiple comparisons for specific pairs of problem means were analyzed. In particular, the following hypotheses were tested:

ANCOVA	Paired Means Hypotheses
Fair-Share	H_0: $FS_{CBI} = FS_{CON}$ H_a: $FS_{CBI} > FS_{CON}$
Center-of-Balance	H_0: $CB_{FSI} = CB_{CON}$ H_a: $CB_{FSI} > CB_{CON}$
Mathematical Concepts	H_0: $MC_{FSI} = MC_{CBI}$ H_a: $MC_{FSI} \neq MC_{CBI}$ H_0: $MC_{FSI} = MC_{CON}$ H_a: $MC_{FSI} > MC_{CON}$ H_0: $MC_{CBI} = MC_{CON}$ H_a: $MC_{CBI} > MC_{CON}$

The top two hypotheses were used to answer research question #1 pertaining to the effect each statistical knowledge conceptualization (i.e. fair-share and center-of-balance) has on the other. These hypotheses were designed to test if knowledge of one statistical conceptualization impacts knowledge of the other conceptualization. The bottom three hypotheses were used to answer research question #2 pertaining to the effect knowledge of each statistical conceptualization has on knowledge of the mathematical concepts related to the arithmetic mean. These hypotheses were designed to compare the effect instruction of fair-share or center-of-balance has on knowledge of the mathematical concepts by comparing posttest scores of mathematical concept problems for each pair of instructional groups. Given that only one a-priori comparison was made from the fair-share and center-of-balance ANCOVA's, the most powerful multiple comparison procedure was a *t*-test with pooled variances. The final ANCOVA, mathematical concepts, produced three a-priori comparisons. The conservative Bonferroni *t* (Dunn's test) was chosen to control the overall familywise error rate for these paired means.

3.5.2 Qualitative Analysis

The verbal protocols and corresponding written problem solutions were qualitatively analyzed in order to interpret, clarify, explain, or otherwise more fully elaborate the results of the statistical analyses. Qualitative analysis of the verbalizations illustrated the nature in which fair-share and center-of-balance were cognitively connected to each other or to the mathematical knowledge domain.

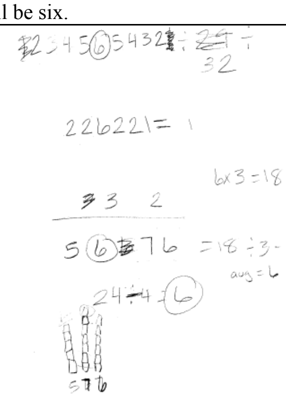
Two areas of interest were identified for qualitative analysis based on results of the pilot study and of previous research (MacCullough, 2007).

1) Qualitatively examining solutions in which participants utilized or connected both statistical conceptualizations of the arithmetic mean, fair-share and center-of-balance, in the problem solving process.
2) Qualitatively investigating connections or relationships of the statistical conceptualizations with mathematical concepts.

Of particular interest was any verbal protocol pertaining to the above described areas that could be reasonably associated with a change in knowledge signified by the statistical analysis. The following two paragraphs provide more detail of the analysis for the two areas of interest identified above.

Instances in verbal protocols that displayed use of both statistical conceptualizations of the arithmetic mean were identified and used to provide descriptions of changes, if any, in participants' knowledge. The protocols and written work were examined for evidence of the cognitive relationship participants exhibited, either implicitly or explicitly, between fair-share and center-of-balance. Figure 3-4 illustrates two examples from the pilot study in which a participant utilized knowledge of both conceptualizations to work toward a solution.

Protocol #1	Code
JO: Well my first guess is zero but does that count as a value? Probably not.	M
JO: Oh I know what value can be added on, the value of the mean. So if your mean is six *(writes down 6 and circles it)* and you add that mean on again you are still going to get a mean of six. Ok, let me do one to make sure. JO: Put a five on either side *(of the circled six)*, a four on either side, a three on each side, and a two and stop there. Let's make sure the mean is six (*adds the numbers)* Twenty-nine, that does not divide evenly. Let's add a one on each side, that's better, thirty-two divided by eleven. Uh, that does not work either. JO: Let's find a number that works equally. Let's get rid of some of these numbers, too many numbers. *(counts numbers)* Nine, I don't like nine. JO: Let's do it differently. Put a six in the middle, a two h MM: Keep talking JO: Ok, That is not going to work. Fourteen divided by five, let's do fifteen divided by six. No. Uh, I do not want to do it that way.	S-CB M-CB M
JO: Ok, let's start an easier way. JO: Six times three is eighteen so let's do three numbers equaling six. *(writes down 2, 3, 2)* Oh, wait, they have to equal eighteen. Three number equaling eighteen, let's do six, a six, no let's do a five *(counts to figure out the seven)*, and a seven. That equals eighteen divided by three, the average equals six. So now I have my three numbers.	M-FS M
JO: Now the mean can be added, now add another six in…twenty-four divided by four is six. So yes, you can add the mean back in so that the data set, I mean the mean does not change.	M
JO: Why does that work. Let me draw a picture. Draw five cubes, seven cubes, and six cubes. In order to make those even we would have to take one from the seven to the six, no, over to the five. All piles will be six. So by adding another pile of six we would not have to move any cubes to make it equal. They will all be six.	S-FS M



Protocol #2	Code
JO: Let's see if the sum of these four numbers, umm, four of these numbers. Ok. Let's draw five blanks. The mean equals 20. The sum of four of those numbers equals seventy-five. So that means if the mean is twenty some number have to be less than twenty so some will have to be greater than twenty.	S-CB
JO: If the sum of four of those numbers is seventy-five then would the fifth number be less than or greater than twenty. Let's see. Let's take some numbers. Twenty, twenty and twenty is sixty, plus to get to seventy-five you would need a fifteen.	M-FS
JO: So that means, let's see, so if you have three twenties and then a fifteen is five less then twenty, then a fifth number needs to be five more than twenty, needs to be twenty-five. Because, it would all even out, because twenty-five is five more than twenty and fifteen is five less than twenty and you want them to all even out to twenty.	M-CB
JO: Let's see if that works… *(adds numbers and divides)* The value of the fifth number would be twenty-five.	M

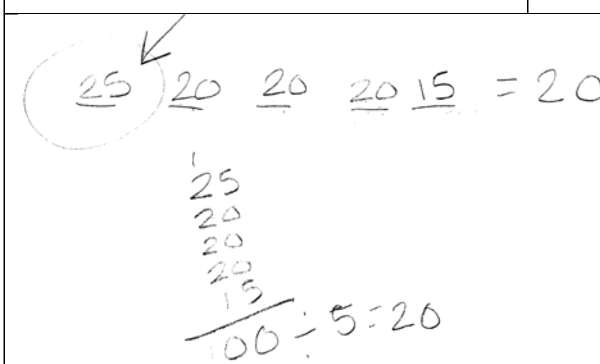


Figure 3-4: Two Conceptualizations in Solution

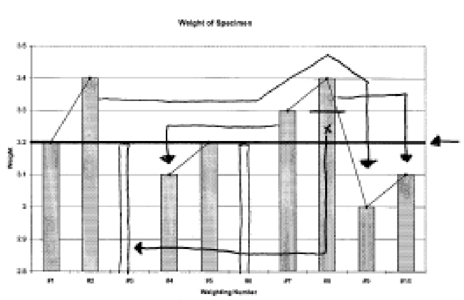
In the first example the participant mistakenly uses the algebraic identity property. She next attempts to use the center-of-balance conceptualization to solve the problem. She correctly identifies the arithmetic mean as a representative number of the data set signified by the point that balances the data set, but misapply the mathematics of the concept while building the data set. Finally, the participant incorporates the conceptualization of fair-share to mathematically build the data set and then to understand the arithmetic mean represents a fair-share allocation of the data. It was the knowledge of the center-of-balance conceptualization that allowed the participant to access the problem; and knowledge of the fair-share conceptualization that allowed the participant to correct mathematical misconceptions. In this case, the participant used both fair-share and center-of-balance conceptualizations, but not necessarily harmoniously in the solution process.

In the second example the participant uses an amalgamation of both statistical conceptualizations in the solution process. She first understands the mean to represent the balancing point of the data; second, she builds a feasible data set using the conceptualization of fair-share; finally, she finds the missing data point using the conceptualization of center-of-balance.

In both examples, knowledge of the two statistical conceptualizations interacted to ultimately reach a solution. In the first example the interaction was as two discrete solution attempts; in the second example the interaction was part of a continuous solution process.

A second area of qualitative investigation focused on the relationships each conceptualization had to mathematical concepts regarding the arithmetic mean. In particular, any mathematical concept associated to the arithmetic mean that is related to both the fair-share and center-of-balance conceptualizations. These relationships were examined to explain or describe any changes in knowledge signified by the statistical analysis. The relationships between the statistical conceptualizations and mathematical concepts were also analyzed as a cognitive link or path between the conceptualizations of fair-share and center-of-balance as proposed by MacCullough (2007). Examples from the pilot study are presented in Figure 3-5

Protocol #3	Code
SC: Whoa! *(pause)* I guess all the graphs should be this line *(referring to the bars and the dark line representing the mean)* *(pause)* MM: Keep Talking SC: Let me work with what I got here. *(SC starts to draw arrows from above the dark line to existing bars below the dark line)* SC: These two *(#2)* will even out this one *(#9)*. This one *(#7)* will even out this one *(#4)* and one of these *(#8)* can go here *(#10)*. SC: Now what? MM: Talk about your thoughts SC: Uh, Uh, I am thinking MM: Verbalize those thoughts SC: I got an extra one *(referring to #8)* I can put here at number three. *(pause)* SC: Oh, each of these *(#3 and #9)* should be four high *(referring to the gridlines)* *(pause)* MM: Keep Talking SC: Four each *(referring to the heights of bar #3 and bar #6)*	S-FS M



Protocol #4	Code
SC: Uh, lets see – I got a $1.38 here, um, $1.30 here, and um, wait, well I guess it fits in here. *(see drawing)* *(pause)* MM: Keep talking	M
SC: Well now I need two higher – this one is eight *(referring to $1.30)* so that's uh, $1.46, and this one is five (referring to $1.35), no three, that's $1.41. SC: How many do I need? Nine SC: So lets see, um $1.37 and $1.39 *(draws in two values)* SC: $1.36 and $1.40 *(draws in two values)* SC: Uh, I already used $1.35 so lets do $1.34 and this one would be uh, that's four, so $1.42. *(draws in two values)* SC: How many is that *(counts values)* Eleven. Oh. Uh. Take these away *(scratches out $1.35 and $1.42)*	S-CB M-CB M
SC: Do you want me to rewrite them? MM: Work until you are finished. SC: I'll make this a little neater *(rewrites numbers in a list)* SC: Done.	

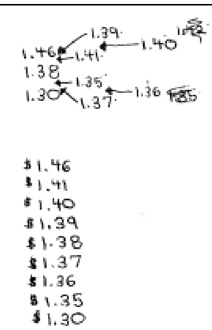


Figure 3-5: Relating Statistical Conceptualizations and Mathematical Concepts in Solution

In protocol #3 the participant uses a fair-share conceptualization as indicated by the phrase "all the graphs should be this line." He then proceed to use the concept 'the sum of the deviations from the mean is zero' to carry out a solution to the problem. In the fourth protocol, the same participant uses a center-of-balance conceptualization and the concept 'the sum of the deviations from the mean is zero' to solve the problem. The participant cognitively connected fair-share and center-of-balance to the same concept, 'the sum of the deviations from the mean is zero'

3.5.3 Data Analysis Summary

The analyses in this study mixed quantitative and qualitative methodologies to identify, substantiate, and more fully describe students' knowledge of fair-share and center-of-balance, and utilized qualitative methods to link these to the mathematical domain of the arithmetic mean. The complete analysis advanced understanding of the cognitive relationships among the conceptualizations and domains of the proposed knowledge structure for the arithmetic mean.

3.6 SUMMARY OF METHODS

The intention of this study was to identify and describe the cognitive relationships between fair-share and center-of-balance as well as the cognitive relationships between these conceptualizations and the mathematical domain of the arithmetic mean. Participants were randomly assigned to one of three groups: (a) instruction of fair-share knowledge, (b) instruction of center-of-balance knowledge, and (c) a control group receiving instruction of general problem solving methods. The methodology of protocol analysis was used to collect, code, and analyze the data. The data was collected via think-aloud verbal protocols and written solutions to arithmetic mean problems. There was a pre- and post- problem solving session with an instructional intervention between the two instruments. The data was coded with a rubric score indicating knowledge of a particular conceptualization (i.e. fair-share and center-of-balance) or knowledge of mathematical concepts. The data was also qualitatively coded to indicate the relationship fair-share and center-of-balance have with the mathematical and statistical domains. The coded data was analyzed using two different methodologies. The first method used statistical analysis to locate any significant relationship between the statistical conceptualizations of fair-share and center-of-balance along with any connections between these and mathematical concepts. The second method included qualitatively analyzing the data to explore solutions in which participants utilized or connected the conceptualizations of the arithmetic mean, fair-share and center-of-balance, or connected either conceptualization with the mathematical and/or statistical domains. The next chapter, chapter four, presents the results of this study.

4.0 RESULTS

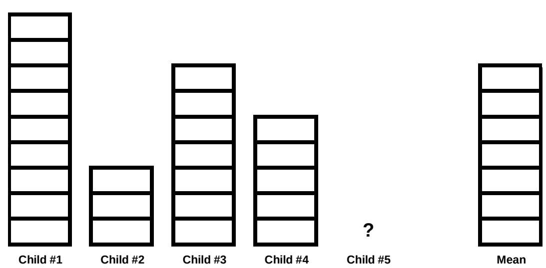
The results of the study are reported in this chapter and are organized into three sections. The first, section 4.1, summarizes the general characteristics of the pretest data. The second and third sections, 4.2 and 4.3, each correspond to one of the two research questions investigated in this study. Section 4.2 describes the cognitive relationship between the fair-share and center-of-balance conceptualizations of the arithmetic mean. Section 4.3 focuses on the relationship between each conceptualization and the mathematical concepts related to the arithmetic mean. In each section, the results of a statistical analysis comparing the relevant posttest scores to a control group are reported, and verbal protocols and written solutions to these problems are examined to further describe the nature of the relationships.

4.1 GENERAL CHARACTERISTICS OF PRETEST DATA

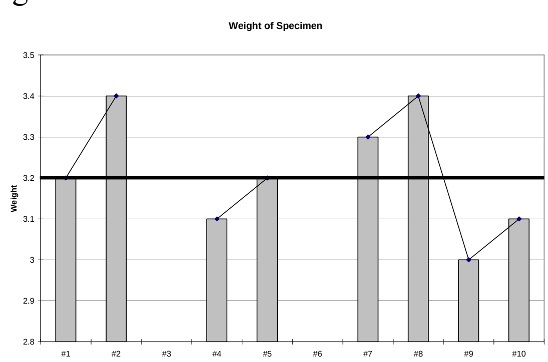
The purpose of this study was to identify and describe any relationship between: (a) the conceptualizations of fair-share and center-of-balance and (b) these conceptualizations and mathematical concepts. Participants completed pre- and post- problem solving sessions and were exposed to one of three instructional interventions, instruction of fair-share knowledge, instruction of center-of-balance knowledge, or instruction of general problem solving methods that was used as a control. The data consisted of think-aloud verbal protocols and written solutions to arithmetic mean problems. The data was coded with a rubric score that indicated knowledge of a particular conceptualization (i.e. fair-share and center-of-balance) or knowledge of mathematical concepts based on a score of zero to three. Mathematically sound solutions that did not use either conceptualization were coded as a "no-score." Qualitative coding was used to indicate if knowledge of any conceptualization, fair-share and/or center-of-balance, was accessed during the solution process.

Table 4-1 depicts the distribution of scores for each of the two fair-share (FS), center-of-balance (CB), and mathematical concept (MC) pretest problems[6]:

FS1: Four children each had a stack of blocks as shown below. When a fifth child sat down with her own set of blocks the mean number of blocks the children had became seven. How many blocks did the fifth child have?



FS2: In a chemistry lab a student weighed a specimen ten times. The results of those weighings are presented in the chart below. The student lost the 3rd and 6th weighings of the specimen after she calculated the mean of the ten weighings to be 3.2 as indicated by the dark line in the graph below. What could have been the values for the 3rd and 6th weighings if the mean is 3.2?



[6] Note: The same problems were used on the pretest and posttest.

CB1: Given three numbers, (a,b,c), and the mean of these numbers is $\bar{x}$. We know that a is 3 greater than $\bar{x}$ and b is 7 greater than $\bar{x}$. How does the value of c relate to $\bar{x}$?

CB2: As a worker in a grocery store you are asked to place price stickers on nine bags of potato chips so that the mean price of the chips is $1.38. You can not price any bag at exactly $1.38. You also must price one bag at $1.30 and a second bag at $1.35. Create the remaining seven price stickers.

MC1: What value can be added to a data set so that the arithmetic mean of the data set does not change? Why?

MC2: The mean of five numbers is 20. The sum of four of these numbers is 75. What is the value of the fifth number?

The table indicates the number of participants receiving a given rubric score and the overall mean for each problem.

Table 4-1: Distribution of Scores for Pretest Data

	Score					
Problem	**3**	**2**	**1**	**0**	**-**	**Mean**
FS1	19	5	2	1	2	2.6
FS2	11	6	8	2	2	2.0
CB1	3	2	10	9	5	1.0
CB2	6	2	4	3	14	1.7
MC1	9	5	4	10	1	1.5
MC2	18	3	2	6	0	2.1

Note. A no-score, indicated by "-", was not included in the calculation of the means. n = 29 for each problem.

Three anomalies were identified in the pretest scores resulting in further qualitative investigation of the verbal protocols and written solutions. First, the overall means of the data seemed to indicate participants had more difficulty solving the center-of-balance problems than they did solving the fair-share problems. Although, it should be noted that participants receiving a no-score correctly solved the problem using an alternative method (i.e. method that did not use fair-share knowledge for fair-share problems or center-of-balance knowledge for center-of-balance problems) but did not count in the calculation of the overall mean. The second discrepancy was an inordinate number of 'zeros' for problems CB1 and MC1 when compared to 'zero' scores on the other problems. In both cases, the 'zero' scores accounted for approximately one-third of the total number of scores on each problem. Third, there were a large number of participants that used an alternative method to solve problem CB2. Nearly half of the responses did not utilize center-of-balance knowledge in the solution. Examining the coded protocols revealed that use/misuse of the arithmetic mean formula was prevalent in solutions linked to all three anomalies. Table 4-2 indicates the overall correct or incorrect use of the arithmetic mean formula in all solution attempts for each of the problems on the pretest.

Table 4-2: Use of the Arithmetic Mean Formula in Pretest Problems

	Applied the Arithmetic Mean Formula		
Problem	**Appropriately**	**Inappropriately**	**Total**
FS1	21	3	24
FS2	11	9	20
CB1	5	5	10
CB2	14	1	15
MC1	2	8	10
MC2	15	6	21

Note: n = 29 for each problem

The remainder of this section further explains how the characteristics of a particular conceptualization (i.e. fair-share or center-of-balance) and/or construct of particular problems affected the role of the arithmetic mean formula.

The discrepancies in the pretest data seem to be related to the propensity of participants to utilize the arithmetic mean formula. Approximately 75% (41 out of 54) of the solutions that used a fair-share conceptualization to solve a fair-share problem suggested some application of the arithmetic mean formula as an integral part of the solution. Conversely, solutions that utilized the center-of-balance conceptualization for center-of-balance problems used the arithmetic mean formula as a final check of the solution and not as an integral part of the solution process. Most all of the alternative methods of solution (i.e. solutions that did not use fair-share knowledge for fair-share problems or center-of-balance knowledge for center-of-balance problems) employed the arithmetic mean formula. The ability to more readily apply the arithmetic mean formula to fair-share problems as opposed to center-of-balance problems may account for the higher scores on fair-share problems. The following example is a typical protocol that used the arithmetic mean formula in a fair-share problem. In this case, the participant used the arithmetic mean formula as a means for calculating the total sum of all data points. The underlined text represents use of the arithmetic mean formula and the highlighted text indicates use of fair-share knowledge.

Pretest FS2

P6: Ok, so first let me get the numbers off of this graph *(writes down values of each weighing)* Ok, so the total for the known weighing is *(uses calculator)* 25.7.

P6: Ok, now I need the total sum of all the weights of the specimens. I know the mean is found by dividing the total weight by the number of specimens so I can find the total weight by multiplying the mean by the number of specimens.

P6: We could possibly infer that all the specimens are equal and weigh 3.2 pounds.

P6: Ok, now to find the weight of the last two specimens subtract the total weight of the known specimens from the 32. That leaves 6.3. One could be higher than that other but if all the weight belonged to one specimen it can weigh no more than 6.3.

$$\begin{array}{r} 3.2 \\ \times\ 10 \\ \hline 32 \end{array}$$

#1: 3.2 #2: 3.4 #3: ? #4: 3.1 #5: 3.2
#6: ? #7: 3.3 #8: 3.4 #9: 3.0 #10: 3.1

Total for known weighings: 25.7

$$\begin{array}{r} 32.0 \\ -\ 25.7 \\ \hline 6.3 \end{array}$$

For problem MC1 there seemed to be a link between using the arithmetic mean formula and the excessive number of ‘zero’ scores. The arithmetic mean formula was not a conducive solution strategy based on the construction of the problem. Similarly, participants found it difficult to apply the formula to the relational data in problem CB1 and were unable to solve the problem using an alternative method to center-of-balance. In contrast, participants’ tendency to correctly use the arithmetic mean formula for problem CB2 indicated it was more applicable to the numeric data. Therefore, a preponderance of ‘no-scores’ were given for a viable alternative method of solution for problem CB2 not involving the center-of-balance conceptualization. The example protocols below illustrate the typical use (or misuse) of the arithmetic mean formula for problems MC1, CB1, and CB2. The underlined text represents use of the arithmetic mean formula.

Pretest MC1

P13: Yea, the number that, take a data set like 3, 4, 5, 6. Add them together equals 18 divide by 4 gives 4.5.

P13: Basically if you add 1 to that, 19 divided by 5 equals, no. Add 2 maybe, umm, 20 divided by 5 equal 4. Try 3…

R: Keep talking

P13: Yea, so take a number equal to or less than the first number and the mean is not going to change much. So 1, 2, or 3 would work. By adding 4 or lower wouldn't change the mean that much. Basically the number would stay the same

$$3, 4, 5, 6 = \frac{18}{4} = 4.5$$

$$1, 2, \text{ or } 3 = 4$$

Pretest CB1

P29: Three numbers, A, B, and C equal the mean x with a little line over it. So A plus B plus C divided by three equals x. A is three greater than x and B is seven greater than x, umm.

R: Keep talking

P29: Well three plus seven is ten so A plus B plus C divided by three might equal 10. So if x with the little line is ten then A can be 13 and B can be 17. Now, how would I solve that?

R: Keep talking.

P29: I am not good at this kind of math. I like the more abstract stuff.

P29: Do over, maybe 20 will work. A plus B plus C divided by three equals 20. A would be 23 and B would be 27 plus C divided by 3 equals 20. I think, no this isn't right I just can't do the math.

$$\frac{A+B+C}{3} = X$$

$$\frac{A+B+C}{3} = 10 \rightarrow \frac{13+17+C}{3} = 10$$

$$\frac{A+B+C}{3} = 20 \rightarrow \frac{23+27+C}{3} = 20$$

Pretest CB2

P12: First I'll set up the equation. The variables a, b, c, d, and e can represent the five numbers. Divide those by five to get the mean, uh, which is 20.

P12: Multiply each side by five which gives us the sum of the variables is now 100. So I'll write that.

P12: Now guess-and-check. I don't know why but I will try counting by fives. I think the 100 and 20 are screaming five at me.

R: Keep talking.

P12: I am just adding every five numbers in my head, I think the second set I thought of will work. 10 plus 15 plus 20 plus 25 plus 30 equals 100. This will only work if four of these equal seventy-five. Give me a second to check this out.

P12: Ok, 10 plus 15 plus 25 plus 30 equals seventy-five. So the value that I did not use, 20, is the fifth number. I don't know if that is what you wanted. I kind of got lucky guessing the numbers.

$$\frac{a+b+c+d+e}{5} = 20$$

$$\frac{a+b+c+d+e}{(5)\,5} = 20(5) \rightarrow 100$$

$$10+15+20+25+30 = 100$$

$$10+15+25+30 = 75$$

$\#$ not used $\rightarrow$ 20

To summarize, the degree to which each pretest problem's construct afforded participants use of the arithmetic mean formula was linked to the problem's scoring. Problems FS1, FS2, CB2, and MC2 were more easily adapted into formula based solutions. Most often the solution was based on the fair-share notion that the sum of the data points is equivalent to the sum of the data points if they were all equal to the mean. In contrast, problems CB1 and MC1 were not easily adapted into a formulaic response. The few participants who scored highly on these problems were able to utilize a center-of-balance approach in their solution method.

4.2 FAIR-SHARE AND CENTER-OF-BALANCE RELATIONSHIP

In this section the results pertaining to research question #1 are examined:

1) How is knowledge of fair-share and center-of-balance cognitively related to one another? In particular,

   a) What effect does instruction of the fair-share conceptualization of the arithmetic mean have on knowledge of the center-of-balance conceptualization?

   b) What effect does instruction of the center-of-balance conceptualization of the arithmetic mean have on knowledge of the fair-share conceptualization?

To answer these questions, written solutions and verbal protocols of pre- and post- test arithmetic mean problems were analyzed both quantitatively and qualitatively to identify how increased knowledge of one conceptualization affected knowledge of the other. The results of these analyses are organized into four parts. First, the results of statistical analyses are reported. Next, two sections explain and otherwise more fully elaborate the statistical analyses of both parts a) and b) for research question #1. Finally, the results of the previous sections are integrated to illustrate the nature of any relationship between the fair-share and center-of-balance conceptualizations

4.2.1 Hypothesis Testing for Research Question #1

The pretest and posttest problem scores served as an indicator of participants' knowledge as it related to either fair-share or center-of-balance with respect to the arithmetic mean. Each problem received a score of zero to three or no-score based on the use and level of understanding of a particular knowledge (i.e. fair-share or center-of-balance) as defined by the rubrics described is section 3.4. An ANCOVA model was used to compare the average posttest scores using the pretest scores as a covariate. Table 4-3 shows the adjusted means for the posttest problems of each group for both the fair-share and center-of-balance problems.

Table 4-3: Adjusted Means for Fair-Share and Center-of-Balance Problems

	Fair-Share Problems		Center-of-Balance Problems	
Instruction Group	**Mean[a]**	**Standard Error**	**Mean[b]**	**Standard Error**
Fair-Share	2.70	.127	2.06	.239
Center-of-Balance	2.73	.126	1.86	.237
Control	2.41	.133	1.32	.267

[a]Pretest covariant mean = 2.28. [b]Pretest covariant mean = 1.22.

Results of the preplanned comparison *t*-test between the fair-share group's center-of-balance mean score (2.06) and the control group's center-of-balance mean score (1.32) indicated the means were significantly different, $t(21) = 2.085$; $p = .026$ (one-tailed). These results indicate center-of-balance problem scores increase with instruction that is focused on the fair-share conceptualization of the arithmetic mean. Similarly, results of the preplanned comparison *t*-test between the center-of-balance group's fair-share mean score (2.73) and the control group's fair-share mean score (2.41) indicated the means were significantly different, $t(25) = 1.747$; $p = 0.043$ (one-tailed). These results indicate fair-share problem scores increase with instruction that is focused on the center-of-balance conceptualization of the arithmetic mean.

4.2.2 Tabular Depiction of Increased Knowledge

Table 4-4 depicts the association between increased scores in fair-share knowledge and center-of-balance knowledge for the group of participants that received fair-share instruction.

Table 4-4: Association of Scores for Fair-Share Instruction Group

		CB Knowledge		
	Increase	NO	YES	TOTAL
FS Knowledge	NO	2 (20%)	2 (20%)	4 (40%)
	YES	2 (20%)	4 (40%)	6 (60%)
	TOTAL	4 (40%)	6 (60%)	10 (100%)

Note: FS denotes fair-share. CB denotes center-of-balance. $n = 10$

Sixty percent (6 out of 10) of the participants receiving fair-share instruction improved their center-of-balance score average. Four participants improved both their CB knowledge and FS knowledge after exposure to fair-share instruction. Two participants improved their CB knowledge, but not their FS knowledge; they had scored highly on the pretest for fair-share problems allowing little or no room for improvement on the posttest.

Table 4-5 depicts the association between increased scores in center-of-balance knowledge and fair-share knowledge for the group of participants that received center-of-balance instruction.

Table 4-5: Association of Scores for Center-of-Balance Instruction Group

		FS Knowledge		
	Increase	NO	YES	TOTAL
CB Knowledge	NO	3 (10%)	2 (40%)	5 (50%)
CB Knowledge	YES	1 (30%)	4 (20%)	5 (50%)
	TOTAL	4 (40%)	6 (60%)	10 (100%)

Note: FS denotes fair-share. CB denotes center-of-balance. *n* = 10

Sixty percent (6 out of 10) of the participants receiving center-of-balance instruction improved their fair-share score average. Four participants improved both their FS knowledge and CB knowledge after exposure to center-of-balance instruction. The two participants who improved their FS knowledge but not their CB knowledge after exposure to center-of-balance instruction used a feasible method of solution on both the pretest and posttest for the center-of-balance problems that did not involve CB knowledge.

The next two sections describe the nature of the increases in knowledge found by the above statistical analyses. Individual problem protocols were coded and analyzed for evidence of fair-share and/or center-of-balance knowledge.

4.2.3 Fair-Share Instruction Impacting Center-of-Balance Knowledge

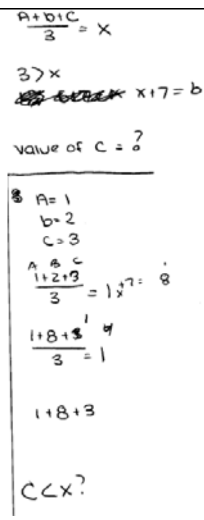
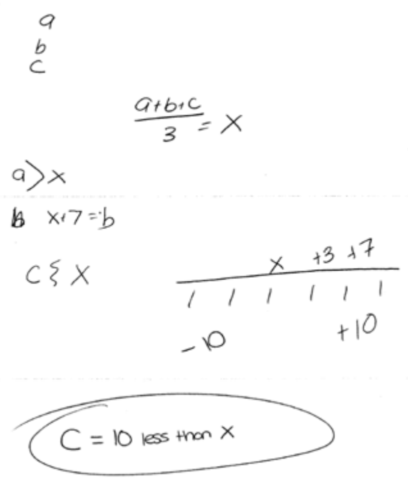
This section explores the solution protocols of the two center-of-balance problems for any connection to the fair-share conceptualization. Of the six participants who improved their center-of-balance knowledge, three improved only on problem CB1, two improved only on problem CB2, and one participant improved on both problems. These seven protocols were qualitatively examined for evidence of fair-share knowledge influencing center-of-balance problem solutions. The impact of the fair-share instruction on center-of-balance knowledge was discernable in five of the seven protocols. The remaining two protocols did not reveal what knowledge of fair-share led to the change in center-of-balance knowledge.

The two solution protocols that did not produce evidence of knowledge transfer originated only in problem CB1. For example, $P4_{fs}$ (participant #4 from the fair-share instruction group), who was unable to solve the problem on the pretest, suggested some knowledge of center-of-balance solving the same problem on the posttest. The response centered around the fact that *c* at least had to be less than $\bar{x}$, but the nature of this change in knowledge between the pretest and posttest was not apparent from the verbal protocol (no written solution was provided). The following comparison from the pretest and posttest protocols indicates the increase in knowledge (bold segments indicate center-of-balance knowledge and 'R' denotes the researcher).

	Pretest CB1		**Posttest CB1**
P4:	I do not know even where to start this problem *(pause)*	P4:	Ok, both these numbers are less than the mean, wait no *b* is bigger. No, ok they are both bigger. **So if these are bigger then this other one has to be smaller.** *(pause)*
R:	Keep talking	R:	Keep talking
P4:	Knowing that they are bigger or smaller does not help if I don't know what the mean is. I can't answer the question without knowing *x*.	P4:	Well, if I don't know what *a* and *b* are I can't find *c*. I don't know, **but *c* has to be smaller than the mean.** It's, it's all I can really tell without knowing the mean.

The five protocols that showed evidence of adapting fair-share knowledge into center-of-balance problems stemmed from both problems CB1 and CB2. In each of the five protocols there was an indication that knowledge learned in the fair-share instruction was transferred to the center-of-balance problems. More specifically, the property 'the sum of the deviations from the arithmetic mean is zero' appeared in the posttest protocols. The following protocols and written work point to the transfer of this knowledge learned in the fair-share context into each of the two center-of-balance problems.

In the first example, $P24_{fs}$ referenced the idea of equal deviations above and below the mean in the posttest protocol, a concept that provided a viable approach to solve the problem not evident in the pretest protocol. While her instruction presented the 'sum of the deviations from the mean is zero' property, it was in the context of fair-share allocations (see Appendix C.1). Although with some hesitation, she correctly transferred the property from fair-share and applied it to the center-of-balance conceptualization.

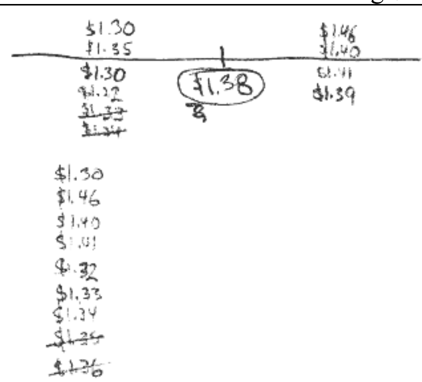
Pretest CB1	Posttest CB1
P24: I am going to write the variable into the mean formula. What else do I know? Umm, I know *a* is 3 greater than *x* and *x* plus 7 is equal to *b*. P24: Ok, what is the value of *c*. I am stuck so let me plug in some values for the variables. 1 for *a*, 2 for *b*, and 3 for *c*. So if I add 7 to 1 I can find out what *b* is. I guess first I need to make 3 greater then x so that's 4. Wait, oh I'm so confused. *(pause)* R: Keep talking P24: I do not know what I am doing. I'm just going to guess that c has to be less than x if x is already three bigger.	P24: Ok, so we can say that *a* plus *b* plus *c* divided by 3 will be *x*. Ok, and we know that *a* is greater than *x* and *b* is 7 greater than *x* so that would mean *x* plus 7 equal *b* and how does the value of *c* relate to *x*. Yea, uh I have no idea. I don't know. P24: I guess if the mean is *x*, well I'll draw a line, like a number line graph so maybe I can use that to figure this out somehow. So *a*, one of the numbers is plus 3 and the other is plus 7; so together they would be plus 10. **So on the left side you have to make it equal to keep the mean at *x* so it would be minus 10. So that keeps it even with *x* in the middle.** So I guess I can say *c* is equal to 10 minus, no 10 less than *x*. That is kind of what it was like on the online module, **if *a* and *b* are 10 greater than *x* then *c* would have to be 10 less than *x*.** That's all I can do with this.
	

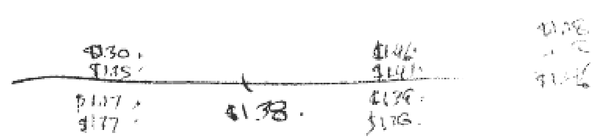
The three participants who improved on problem CB2 also used the property, 'the sum of the deviations from the mean is zero', to transfer their fair-share instruction to the center-of-balance problem. The next protocol characterized two of the three participant's responses. In this case the participant, P14$_{fs}$, was unable to solve the problem on the pretest using a guess-and-check method that adjusted the data points based on the calculated mean; then correctly and efficiently solved the problem on the posttest using the center-of-balance conceptualization. He used the two given amounts, \$1.30 and \$1.35, as the only data points below the mean and adjusts the seven missing values above the mean to equalize the deviations.

Pretest CB2	Posttest CB2
P14: I know that the average is found by adding up all the values and dividing by the number that you have. In this case I am going to need nine numbers, uh, I already have two. *(pause)* R: Keep talking P14: I'll just pick numbers in a reasonable range of \$1.38 without using \$1.38. *(writes down the two given values and seven values; uses calculator to find the mean)* P14: \$1.51, that is way too high so I'll tone down the higher numbers I used. *(writes down the two given values and seven values; uses calculator to find the mean)* P14: Almost, too low. I'll try again with slightly higher numbers. *(writes down the two given values and seven values; uses calculator to find the mean)* P14: So they still need to be a little higher, I would just keep going until I get \$1.38. You get the idea.	P14: So the mean is \$1.38 and I can't use \$1.38 and I got \$1.30 and a \$1.35 and I need seven more so...*(draws seven blank lines)* P14: **So, if the average is \$1.38 then I need the same above the mean that is below \$1.38. I already have two numbers below the mean so I need numbers bigger than the mean.** *(writes down \$1.39 in each of the blanks)* R: Keep talking P14: See if \$1.39 works. **So we have 8 and 3 below the mean which is 11, right, yea, and *(counts one for each \$1.39)* seven above the mean. Oh, that is not enough, we need a lot larger numbers. So let's change these last four to \$1.40; that will give us four more. *(counts the deviations above the mean)* That is eleven on both sides of the mean so that should be right.** P14: Let me check by dividing *(uses calculator to check solution)*
1.3 + 1.35 + 1.40 + 1.33 + 1.39 + 1.37 + 1.35 + 1.36 + 1.39 = $\frac{13.61}{9}$ = 1.51 1.3 + 1.35 + 1.35 + 1.33 + 1.37 + 1.36 + 1.3 + 1.39 + 1.3 = $\frac{12.05}{9}$ = 1.34 1.3 + 1.35 + 1.39 + 1.40 + 1.3 + 1.32 + 1.33 + 1.35 + 1.37 = $\frac{12.11}{9}$ = 1.35	mean = 1.38 1.3 1.35 | 1.39 1.39 1.39 1.40 1.40 1.40 1.48 $\frac{12.42}{9}$ = 1.38

The next protocol, from participant $P1_{fs}$, also shows how the property, 'the sum of the deviations from the mean is zero,' was transferred from fair-share instruction to a center-of-balance problem. The property, learned in the context of the fair-share conceptualization, was utilized to correct a misconception the participant had concerning the arithmetic mean as a center-of-balance. In the pretest protocol, $P1_{fs}$ mistakenly "balanced" the mean in the center of the data as a *median*, based on *number* of data points above and below the mean; rather than as the *arithmetic mean*, based on the *deviations* above and below the mean. In the posttest protocol, he corrected his initial error remembering his mistake on the pretest, "The number line thingy was right but not counting spaces messed me up."

	Pretest CB2		**Posttest CB2**
P1:	So a $1.38 is **the average so it, like on a number line, could be in the middle.** So I will draw that here. These two numbers, the $1.30 and $1.35 go on this side of the average.	P1:	Oh yea, last time I really messed this up. **The number line thingy was right but not counting spaces messed me up.** So let's try it again.
P1:	Now what do I need to do? *(pause)*	P1:	**$1.38 in the middle, $1.30 here, and $1.35 here. Now I'll do it right this time. I have to count the spaces between these and the average. So $1.30 is *(counts to eight)* 8 less, meaning we need one 8 more. $1.35 is 3 less so $1.41 is three more.** Ok, that takes care of four of them. *(pause)*
R:	Keep talking		
P1:	Well, I'm just going to pick some numbers between $1.30 and say, umm, **well that is eight below so let's do $1.46.** *(writes down numbers)*	R:	Keep talking
P1:	Ok, now, umm, no we have two numbers already so I only need seven. *(erases two numbers)*	P1:	I guess I can make the rest up, right?
P1:	Ok, how do these fit in? *(places numbers on the number line)*	R:	I can't help you.
R:	Keep talking	P1:	**Well I'll make it easy, $1.37 and 1.39 and another $1.37 and $1.39.** That's nine. *(mistakenly counts the $1.38)*
P1:	That is not going to work, **I have too many numbers below the mean. Just change one of these to a bigger than average number and it should work.**		
P1:	Wait, we have to use $1.38 or it won't work. I do not think it can be done without using $1.38.		





Thirteen of the 20 protocols (2 problems for each of 10 participants) demonstrated no indication of change in center-of-balance knowledge between the pretest and posttest. Over half (7 out of 13) of these were correctly solved on the pretest either with a solution based on the center-of-balance conceptualization or with a solution based on an alternative method. In each of these cases the posttest protocol mirrored the solution on the pretest problem.[7] The remaining six protocols equally exhibited little or no center-of-balance knowledge on either the pretest or posttest.

To summarize this section, significant differences in center-of-balance knowledge were observed between participants given fair-share instruction and a control group. The nature of the increase in knowledge appeared to stem from the ability to transfer knowledge gained in fair-share instruction to center-of-balance problems. The concept, 'the sum of the deviations from the mean is zero,' was perceptible in every protocol that indicated evidence of fair-share knowledge impacting center-of-balance knowledge.

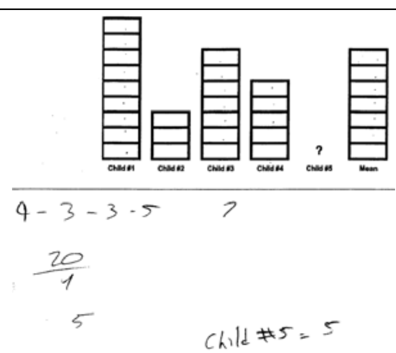
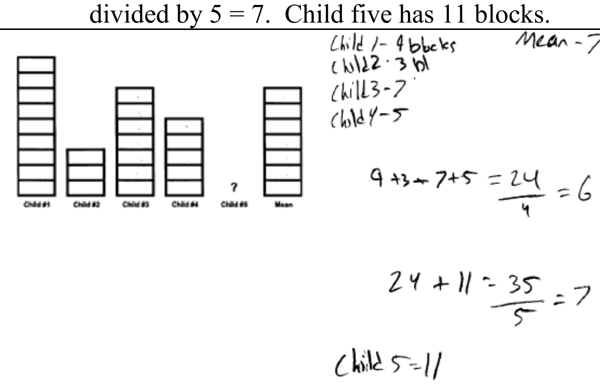
4.2.4 Center-of-Balance Instruction Impacting Fair-Share Knowledge

This section explores the solution protocols of the two fair-share problems for any connection to the center-of-balance conceptualization. Of the six participants who improved their fair-share knowledge, three improved only on problem FS1, two improved only on problem FS2, and no participant improved on both problems.[8] The five protocols that showed improved scores could be categorized into two groupings based on what seemed to be associated with that improvement: (a) those who initially failed to consider all data points on the pretest, but included

[7] In one case, not counted in the 13 referenced here, a participant used a viable alternative method for the pretest and then a correct center-of-balance method on the posttest. See $P14_{fs}$ protocol in this section.
[8] One participant used a viable method on the posttest that did not utilize fair-share knowledge for a problem scored a one on the pretest; thus their overall average increased for the two problems without increasing any one score.

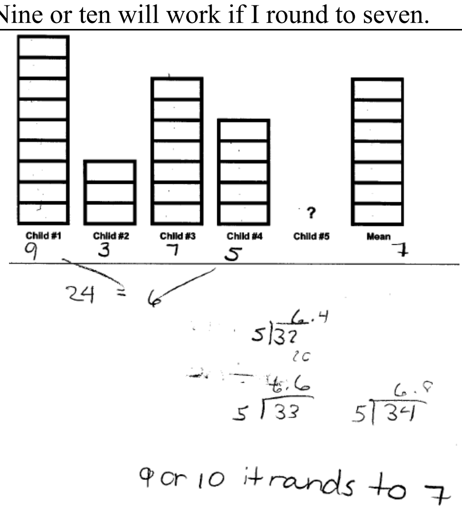
all data points on the posttest and (b) those that used knowledge of deviations from the mean on the posttest.

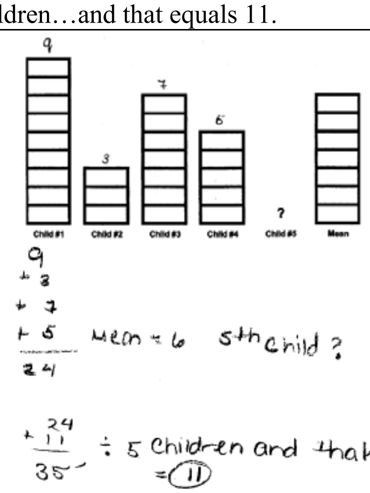
A common improvement on both center-of-balance problems after fair-share instruction was the inclusion of all data points when determining the mean. Initially, three participants failed to account for all the data in their solution strategies. The pretest protocol below illustrates how participant $P13_{cb}$ failed to include the missing data point in the denominator when calculating the mean of all data points. On the posttest, $P13_{cb}$ added to the pretest protocol by including the missing data point in the total. $P13_{cb}$ realized the sum of the given data points is equivalent to the sum if each child had the mean number of blocks. The highlighted areas indicate use of fair-share knowledge.

Pretest FS1	**Posttest FS1**
P13: I am going to take the four children and write down the number *(counts each stack and writes down the numbers, counts seven for the third child but writes down 3)* I know the mean is seven. If I add the four original children I get 20 divided by the original amount four is five.	P13: Child #1 nine blocks, child #2 three blocks, child #3 seven, child #4 five and the mean is seven. P13: If I add the numbers I know and divide by the number of kids I will reach the original average. Ok that's 9 + 3 + 7 + 5 = 24 divided by 4 = 6.
P13: Basically, to get a mean of five I would need to add in another five. It's like five plus five, five times. Child five equals five	P13: I know the total amount of blocks I need is 35. You want me to explain that. R: Yes P13: Basically, the total equal five times the mean since the total divided by the number of kids equal seven. So five times seven is 35. P13: Yea, I need 11 more blocks so 24 + 11 = 35 divided by 5 = 7. Child five has 11 blocks.
	

A second grouping of protocols also revealed evidence that center-of-balance instruction helped increase knowledge on fair-share problems. Three of the five protocols that demonstrated an increase in fair-share knowledge included use of the property, ‘the sum of the deviations from the arithmetic mean is zero,’ in the posttest but not in the pretest. In the example below, $P27_{cb}$ unsuccessfully used a formula-based guess-and-check method on the pretest; then successfully incorporated the idea of equal deviations from the mean in the posttest protocol. In this case, the concept, ‘the sum of the deviations from the arithmetic mean is zero,’ was learned as it related to center-of-balance (see Appendix C.2). $P27_{cb}$ successfully adapted knowledge of this concept from center-of-balance instruction and applied it as a fair-share conceptualization.

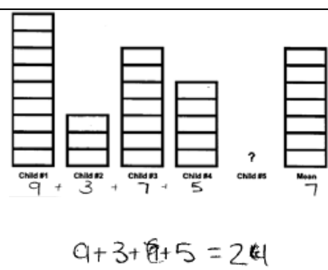
Pretest FS1	Posttest FS1
P27: Child #1 has nine blocks, child #2 has three blocks, child #3 has seven blocks, child #4 has five blocks, and the mean has seven blocks. If I add these up *(the first four children)*…24. The mean equals six. *(pause)* R: Keep talking. P27: I am going to pick random numbers to see if they will fit and end up with a total mean to be seven. Six would give me 30 divided by five is six. No. *(erases work)* R: Please do not erase any work. P27: If I add eight… that's 32 divided by five…6.4. I'll try nine. Can I use my calculator? R: Yes, but keep talking. P27: That's 33 divided by five…6.6. Maybe 10. 34 divided by 5…6.8. P27: Nine or ten will work if I round to seven.	P27: Child #1 has nine blocks, child #2 has three blocks, child #3 has seven blocks, and child #4 has five blocks. If I add the four children's blocks up I get 24 and the mean for these four would be six. R: Keep talking P27: But the sixth child is unknown and the mean all together is seven so the six doesn't work. R: Keep talking P27: That means the mean, huh, of the first four is one less than the all together mean. R: Keep talking P27: So I need to add one to each of their means. That's four all together and… the mean of child #5 would be seven. Should I add those? So four plus seven is 11. P27: So 24 plus 11 is 35 divided by five children…and that equals 11.

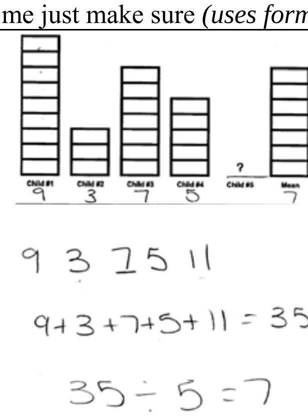




One protocol had both of the characteristics evidenced in improved fair-share problems in the center-of-balance instruction group. That is, it failed to consider all the data points in the pretest and used knowledge of deviations from the mean on the posttest. In this example, it is clear that knowledge pertaining to the center-of-balance conceptualization of the arithmetic mean (indicated by bold in the posttest protocol) was used in conjunction with fair-share knowledge (indicated by highlight in the pretest and posttest protocols) to reach a viable solution.

Pretest FS1	**Posttest FS1**
P17: There are five children and the mean is seven. I do not exactly remember what the mean is but I think that it is the numbers combined and divided by the number. *(pause)* R: Keep talking. P17: So child three has seven blocks, the same number as the mean. If everyone had seven then I think the mean should be seven. Well, I guess they don't and that is why you gave us the problem. P17: I think I can make the ones that are not seven, child #1, child #2, and let's see, child #4 average seven and that would work. I don't know what else to do so I'll do that. I got three kids so 21 divided by three will give me the seven. Nine plus three plus seven plus five equals 24. Umm, wait, oh, I don't need the seven. So that's seventeen. *(mentally adds 9+3+5)* So I need four more to get 21. I think child #5 might have 4 blocks.	P17: Well I look at the blocks and I see that child #1 has nine blocks, child #2 has three, child #3 has seven, child 4 has five, and child #6 is blank and we I need a mean of seven. So the problem is how many blocks does he have to make the mean seven, so… P17: Well, child #3 has seven blocks and that is the same as the mean. Looking at my numbers I see two odd numbers, the three and the five that are less then the seven and one odd number that is bigger, the nine. So there is a pattern, every two odd numbers less than and bigger than the mean. The pattern says that child #5 should have 11 blocks. P17: So everything has to be like they were all seven. The seven is seven. Now, umm… *(pause)* P17: **If I combine these two** *(nine and five)* **I get 14 divided by two is seven so good. Now combine these two** *(three and eleven)*, **14 again.** P17: So let me just make sure *(uses formula to check)*





Eleven of the 20 protocols (2 problems for each of 10 participants) did not demonstrate any indication of change in fair-share knowledge after center-of-balance instruction. Over 80% (9 out of 11) of these protocols were correctly solved on the pretest either with a solution based on the fair-share conceptualization or with a solution based on an alternative method. In each of these cases the posttest protocol closely resembled the solution on the pretest. In two other protocols, problem FS1 was correctly solved on the pretest with an alternative method (an algebraic approach based on the arithmetic mean formula and the missing data point as a variable), and then with a correct center-of-balance conceptualization on the posttest. Conversely, in one case, problem FS2 was successfully solved using a center-of-balance conceptualization on the pretest and then with an alternative method (an algebraic approach) on the posttest.

To summarize this section, significant differences in fair-share knowledge were observed between participants given center-of-balance instruction and a control group. The nature of the increase in fair-share knowledge after the center-of-balance instruction was characterized by two themes: (a) an understanding that all data points, including the missing data points, were relevant and (b) the ability to adapt knowledge concerning deviations from the mean gained in center-of-balance instruction to fair-share problems. Every protocol that indicated center-of-balance knowledge impacted fair-share knowledge included one or both of these themes.

4.2.5 Integration of Fair-Share and Center-of-Balance Results

Rarely did participants directly use the specific fair-share or center-of-balance knowledge that they learned in the instructional modules on posttest problems of the other conceptualization. For example, participants did not apply a balance model learned in the center-of-balance instruction to a fair-share posttest problem; nor did they apply reallocation of the original data set learned in the fair-share instruction to solve center-of-balance problems. Rather, they used a manifestation of that knowledge signified by the concept, ‘the sum of the deviations from the arithmetic mean is zero.’ Participants were able to transfer knowledge of this concept learned in a fair-share context to center-of-balance problems, and similarly, transfer knowledge of this concept learned in a center-of-balance context to fair-share problems.

Examples of using both center-of-balance and fair-share knowledge in the same problem occurred in only three protocols. In each case, the protocols indicated use of both fair-share and center-of-balance knowledge integrally connected in the solution process. The previously presented posttest protocol of $P17_{cb}$ is one such case. The statement, “So everything has to be like they were all seven” indicated a fair-share conceptualization of the data. Next, $P17_{cb}$ used this idea to balance the data points, “Seven is seven. If I combine these two *(nine and five)* I get 14, divided by two is seven, so good. Now combine these two *(three and eleven)*, 14 again.” A second protocol (from problem FS2) that exemplified a relationship between fair-share and center-of-balance utilized an identically successful method of solution on both the pretest and posttest. Initially, $P15_{cb}$ thought of the aggregate total of all data points, a fair-share conceptualization, and related this knowledge to deviations from the mean. “Well, the values of the 3^{rd} and 6^{th} weighing would need to make the total value 32. They can be more or less than the mean, which ever they need to be to make the distance equal.” $P15_{cb}$ then took the concept of deviations from the mean and transferred it to a center-of-balance conceptualization. “I am going to figure out how much I have more than the mean and how much less. When I add those up I know the positives and negatives should balance out.” The problem was solved within the context of center-of-balance and then transferred back to fair-share to check the solution. “See it that all adds up to 32…and 32 divided by 10 equals 3.2.” Knowledge pertaining to the center-of-balance conceptualization of the arithmetic mean is indicated by bold and knowledge pertaining to the fair-share conceptualization is indicated by highlight.

FS2

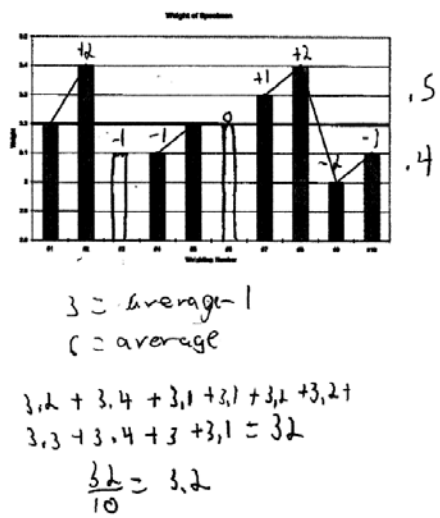
P15: Well, the values of the 3rd and 6th weighing would need to make the total value 32. They can be more or less than the mean, which ever they need to be to make the distance equal. They might also be the mean, let's see.

P15: **I am going to figure out how much I have more than the mean and how much less. When I add those up I know the positives and negatives should balance out. If they don't then #3 or #6 will make it work.**

P15: #1 and #5 are the mean so they don't count. This one is +.2, this is +.1, and this is +.2 *(referring to weighings #2, #7, and #8)*. And down here it's .1 less, .2 less and .1 less *(referring to weighings #4, #9, and #10)* So the value greater than is .5 and the value less than is .4.

P15: So I know that, umm, #3 and #6 added together need to be .1 less than the mean so that it all works out. So value #3 could be .1 less and value #6 could be exactly the average. You could also change it so that any combination is .1 less than the average

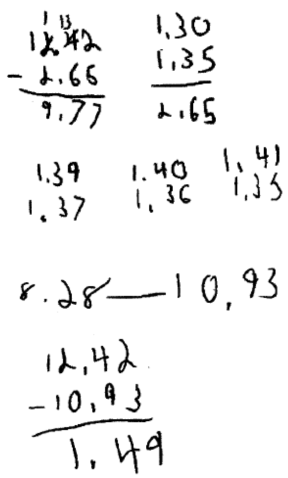
P15: See it that all adds up to 32…and 32 divided by 10 equals 3.2



The same participant, $P15_{cb}$, who used both center-of-balance and fair-share knowledge in a fair-share problem also utilized both knowledges in a center-of-balance problem. The participant used the same solution process for both the pretest and the posttest for problem CB2. $P15_{cb}$ first used fair-share knowledge to note that all nine bags would total $12.42, and that $9.77 was the total of the remaining seven bags once the two known prices were subtracted. He then used center-of-balance knowledge to calculate several of the remaining unknown prices by picking values equal distant above and below the mean. Finally, the last price was found by calculating what remained of the total.

CB2

P15:	In order for the average value to be \$1.38 the price of all the potato chips if I add them has to be the mean times nine. So that is \$12.42.
P15:	I know I have these two, \$1.30 and \$1.35, *(uses calculator)* \$2.65. So the total of all the values of the chips minus those two, *(uses calculator)* \$9.77.
P15:	**I am going to choose numbers on either side of \$1.38 to make it easy to figure out the mean. \$1.39 and \$1.37; \$1.40 and \$1.36; \$1.41 and \$1.35.**
P15:	Altogether that is, *(uses calculator)* \$8.28. The \$1.30 and \$1.35 make it, *(uses calculator)* \$10.93.
P15:	\$12.42 minus \$10.93, *(uses calculator)* \$1.49 for the last bag.



A common solution structure emerged from each of the three cases that integrated both conceptualizations into one protocol. The participants first framed the solution within the context of the fair-share conceptualization. In the first case, $P17_{cb}$ used fair-share to note the mean would remain at seven if all the data points were equal to seven. In the second and third cases, $P15_{cb}$ used fair-share to find aggregate totals to form a basis for the solutions. Once the problems were framed within the fair-share conceptualization the participants then used the idea of center-of-balance to carry out the solution process. $P17_{cb}$ (case #1 above) balanced pairs of data points around the mean for problem FS1. Similarly, $P15_{cb}$ (case #3 above) balanced pairs of data points to solve problem CB2. A slightly different center-of-balance approach was employed by $P15_{cb}$ to solve problem FS2 (case #2 above). In this case, the amassed deviations above and below the mean were first calculated, and then the missing data points were chosen to balance the entire data set.

While most evidence indicates participants did not directly apply learned knowledge of the fair-share or center-of-balance conceptualization into posttest problems based on the opposite conceptualization (cf. $P17_{cb}$); evidence does indicate knowledge learned in one conceptualization was transferred to the other. In particular, knowledge that the 'sum of the deviations from the arithmetic mean is zero' appeared in the posttest protocols for both the fair-share and center-of-balance instructional groups. About sixty-six percent (8 out of 12) of the participants that showed improved scores transferred knowledge of this idea learned in one context (i.e. fair-share or center-of-balance) to the other.

4.3 CONCEPTUALIZATIONS AND MATHEMATICAL CONCEPTS

In this section the results pertaining to research question #2 are examined:

2) How is knowledge of fair-share and center-of-balance cognitively related to the mathematical domain? In particular,

   a) What effect does instruction of the fair-share conceptualization of the arithmetic mean have on knowledge of mathematical concepts associated with the arithmetic mean?

   b) What effect does instruction of the center-of-balance conceptualization of the arithmetic mean have on knowledge of mathematical concepts associated with the arithmetic mean?

To answer these questions, written solutions and verbal protocols of pre- and post- test arithmetic mean problems were analyzed both quantitatively and qualitatively in order to identify how increased knowledge of fair-share or center-of-balance affected knowledge of particular mathematical concepts related to the arithmetic mean. The results of these analyses are organized into four sections. First, the results of statistical analysis are reported. The next two sections detail how each conceptualization impacts mathematical concept knowledge. Last, the nature of the relationships between the conceptualizations of fair-share and center-of-balance and the mathematical concepts related to the arithmetic mean are summarized.

4.3.1 Hypothesis Testing for Research Question #2

The pretest and posttest scores for the mathematical concept problems served as an indicator of participants' capacity to integrate mathematical knowledge with other mathematical concepts or with the conceptualizations of fair-share and center-of-balance. Each problem received a score of zero to three based on its correctness and use of sound mathematical ideas, or "no-score" if the correct solution path was ambiguous. An ANCOVA model was used to compare the average posttest scores using the pretest scores as a covariate. Three hypotheses were tested based on the mathematical concept problems' posttest scores: (a) differences between mathematical concept posttest scores for the group that received fair-share instruction and the group that received center-of-balance instruction, (b) differences in mathematical concept posttest scores between the group that received fair-share instruction and a control group, and (c) differences in mathematical concept posttest scores between the group that received center-of-balance instruction and a control group. Table 4-6 shows the adjusted means for the mathematical concept posttest problems of each group.

Table 4-6: Adjusted Means for Mathematical Concept Problems

Instruction Group	Mathematical Concept Problems: Mean[a]	Standard Error
Fair-Share	2.35	.167
Center-of-Balance	2.52	.166
Control	1.92	.176

[a]Pretest covariant mean = 1.83

Results of the Bonferroni *t* (Dunn's test) between the fair-share group's mathematical concept mean score (2.35) and center-of-balance group's mathematical concept mean score (2.52) indicated the means were not significantly different, $t'(25) = 0.72$; $p \approx 1.00$. Results of the Bonferroni *t*-test between the fair-share group's mathematical concept mean score (2.35) and the control group's mathematical concept mean score (1.92) indicated the means were not significantly different, $t'(25) = 1.86$; $p = .13$ (one-tailed). Results of the Bonferroni *t*-test between the center-of-balance group's mathematical concept mean score (2.52) and the control group's mathematical concept mean score (1.92) indicated the means were significantly different, $t'(25) = 2.58$; $p = .029$ (one-tailed). This result indicates mathematical concept scores increase with instruction that is focused on the center-of-balance conceptualization of the arithmetic mean. The following sections describe the nature of the results found by the above statistical analysis.

4.3.2 Fair-Share Instruction Impacting Mathematical Concepts

This section explores the solution protocols of the two mathematical concept problems for any connection to the fair-share conceptualization. Fifty percent (5 out of 10) of the participants in the fair-share group advanced their mathematical concept knowledge after fair-share based instruction. Each of the five participants improved their score on one of the two mathematical concept problems. Therefore, twenty-five percent (5 out of 20) of the scores for the mathematical concept posttest problems were improved upon from the pretest. Of the remaining fifteen scores, ten were perfect (i.e. rubric score of 3 out of 3) on the pretest and posttest meaning there was no potential for improvement. Hence, fifty percent (5 out of 10) of the problems that had "improvable" scores on the pretest were bettered on the posttest. Of these, two improvements were on problem MC1 and the remaining three improvements were on problem MC2.

The protocols of both participants in the fair-share group who increased their score on MC1 did not indicate what new knowledge was responsible for the change. In one case, the participant confidently and matter-of-factly stated the correct solution on the posttest after providing a hesitant and incomplete solution of the pretest. Below is an example of that pretest and posttest protocol.

Pretest MC1	**Posttest MC1**
P26: I don't know. It would depend on the data set. Every data set would be different.	P26: The mean, yea, the arithmetic mean. I'll write it out.
	The mean of the data set could be added without changing the original arithmetic mean because the mean is neither higher or lower than the arithmetic mean so it would have no power to change it.

The three participants who increased their scores for problem MC2 each provided protocols that revealed the knowledge they used to solve the problems. One of them was unable to solve the problem on the pretest using a guess-and-check method, but correctly solved the problem on the posttest using an algebraic approach based solely on the arithmetic mean formula. There was no evidence of fair-share knowledge in the solution. The remaining two participants utilized fair-share knowledge in the posttest that was not evident in the pretest. In the following protocol, P18$_{fs}$ struggled toward a solution on the pretest using the arithmetic mean formula. On the posttest, she used the sense of fair-share to find the aggregate total of all five data points and then subtracted the sum of the first four to calculate the fifth data point. The addition of fair-share knowledge into the problem solution on the posttest provided a solution path that was more coherent and mathematically sound then the chaotic nature of the pretest solution.

Pretest MC2	**Posttest MC2**
P18: To find the mean of the first four numbers I will divide by four. *(pause)*	P18: Ok, five numbers and the mean is twenty so 20x5 = 100 which is the total of all five numbers with the missing one.
R: Keep talking.	P18: The total minus the four that we know will give us the missing number. 100-25 = 75. That has to be the missing number.
P18: Eighteen point seven five. Then 18.75 plus 75 equals (*uses calculator*) is 93.75. 93.75 divided by 5 is 18.75, huh. *(pause)*	
R: Keep talking.	
P18: Ok, if the mean is 20…75 plus 20 is 95. 95 divided by 5 is 19. *(pause)*	
R: Keep talking.	
P18: 75 plus 75 is 150. 150 divided by 5 is 30. 75 plus 30 is 105 divided by 5 is 21.	
P18: So it is between twenty and thirty.	
$75 \div 4 = 18.75$	$20 \times 5 = 100$ $100 - 75 = \boxed{25}$ #5

Of the ten perfect scores on the pretest, four were scored on problem MC1 and six were scored on problem MC2. The correct solutions for problem MC1 fit into one of two categories: induction (3) or center-of-balance (1). There was no evidence of fair-share knowledge in any solution for problem MC1 in the fair-share group. The correct solutions for problem MC2 fit into one of two categories: fair-share (4) or algebraic (2). Each of the four fair-share based solutions was similar to $P18_{fs}$'s posttest protocol previously presented.

To summarize this section, no significant difference in mathematical concept knowledge was statistically calculated between participants given fair-share instruction and a control group. However, several protocols, particularly those of problem MC2, did suggest using knowledge of the fair-share conceptualization lead to a correct solution or improved scores.

4.3.3 Center-of-Balance Instruction Impacting Mathematical Concepts

This section explores the solution protocols of the two mathematical concept problems for any connection to the center-of-balance conceptualization. Eighty percent (8 out of 10) of the participants in the center-of-balance group advanced their mathematical concept knowledge after center-of-balance based instruction. Four of the participants improved only on problem MC1, three participants improved only on MC2, and one participant improved on both problems. Therefore, forty-five percent (9 out of 20) of the scores for the mathematical concept posttest problems were improved upon from the pretest. Of the remaining eleven scores, nine were perfect (i.e. rubric score of 3 out of 3) on the pretest and posttest meaning there was no potential for improvement. Hence, about eighty-two percent (9 out of 11) of the mathematical concept problems that had "improvable" scores on the pretest were bettered on the posttest after center-of-balance instruction.

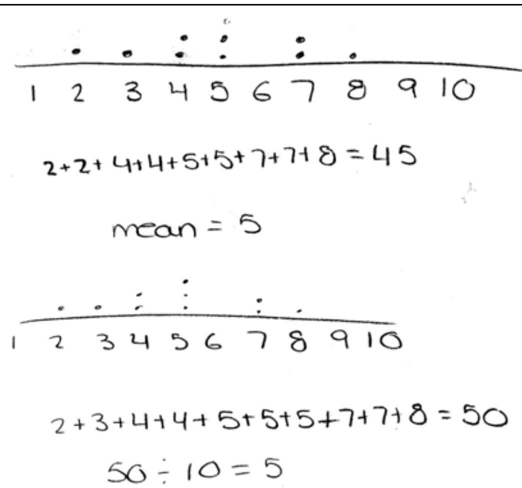
One of the five participants that improved on problem MC1 offered no indication as to why their answers differed on the pretest and posttest. A second participant used an algebraic method based on the arithmetic mean formula to improve his initial incorrect logic. $P3_{cb}$

incorrectly answered zero on the pretest. He erroneously assigned the algebraic properties of zero in addition and multiplication to the arithmetic mean. On the posttest, he used algebra and the arithmetic mean formula to first disprove his initial answer and then found the correct solution, but only for his arbitrary data set. He never stated that the solution was the mean for all data sets.

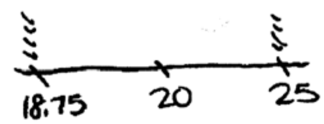
Pretest MC1	**Posttest MC1**
P3: The number zero can be added to the data set and the mean will not change.	P3: Zero can be added to the data set not to change the mean.
P3: It will not change because zero is almost like an invisible number. Whether you multiply or add zero to other numbers you will either get zero or the number that you are adding the number zero with.	P3: I will give an example. One, two, three, four, five. These numbers add up to 15 which divided by five is three P3: 15 plus x can be divided by three, no divided by six, will equal three. P3: Let x equal zero. No. uh, wait, no. *(pause)* R: Keep talking P3: Solve the equation for x. *(algebraically solves equation)* umm, three. P3: Zero does not work. The answer is three.
	1 2 3 4 5 = 15 ÷ 5 = 3 $\frac{15+x}{6} = 3$ $15 \overset{-15}{+} x = 18^{-15}$ $x = 3$

The remaining three participants who improved on problem MC1 indicated new use or improved use of the center-of-balance conceptualization from the pretest to the posttest. For example, $P17_{cb}$ did not have a strategy to solve the problem on the pretest but used a center-of-balance approach on the posttest to correctly solve the problem. On the posttest she constructed a data set on a number line with an arbitrary mean (five) using center-of-balance to maintain the mean. She then pointed out, "the only way to add one dot and not make it uneven is to put the dot on the mean." In this case, the center-of-balance conceptualization was utilized to set-up a strategy and to deduce a valid solution.

Pretest MC1	Posttest MC1
P17: I'm not sure what the data set is. It would depend of what the data set looked like. I could use the equation to find the answer if there was data. *(pause)* R: Keep talking. P17: Like I said, I could try a bunch of numbers to see what worked if I had the data set. *(pause)* R: Keep talking. P17: Should I make one up. R: Do what you need to answer the question. P17: It wouldn't help since it would change for every group. It is different for every group	P17: We did the module online so I figure I can use that to solve this problem. Since I did this problem last I realize that **you can set up a balance line to show the answer.** (*draws number line*) P17: Suppose the mean is five, I'll put three dots, no two, on the five. **Now if I put one on the three I put one on the seven. I can also put another on the seven, but instead of the three I can put two on the four to make it even. And I'll put one on the eight and on the two.** P17: *(adds the numbers)* So that's 45 dots divided by nine is five, good. P17: Now, **the only way to add one dot and not make it uneven is to put the dot on the mean.** *(redraws number line with another dot on the five)* P17: *(adds the numbers)* 50 dots. This time divided by ten and my mean is still five.



Problem MC2 had four improved scores. Of these, two participants modified a fair-share approach to solving the problem, one participant successfully used an algebraic approach, and one participant used a center-of-balance approach. $P28_{cb}$ used the arithmetic mean formula on four data points and was unable to find a successful solution strategy on the pretest. On the posttest, she again used the arithmetic mean formula but this time combined it with a balance model. $P28_{cb}$ applied knowledge learned regarding the center-of-balance conceptualization, along with her initial strategy that focused on the arithmetic mean formula, to further her ability to solve the problem.

Pretest MC2	**Posttest MC2**
P28: The mean of the numbers is 75 divided 4 equal 18.75. That is close enough to 20 to be part of the data set.	P28: **The mean is in the middle.** Twenty. Four numbers equal 75. 75 divided by 4 is 18.75. Four of them. P28: **On the other side of twenty I need four more.** *(uses calculator).* 20 minus 18.75 equal 1.25 times 4 equal 5 plus 20 equal 25. *(draws four slashes above 25)* P28: Twenty-five will work
$\frac{75}{4} = 18.75$	

To summarize this section, the difference in means between the center-of-balance group's mathematical concept score and the control group's mathematical concept was significant. About forty-four percent (4 out of 9) of score improvements on the mathematical concept problems were associated with increased center-of-balance knowledge. Center-of-balance knowledge appeared in fifty percent (5 out of 10) of the CB1 posttest protocols and in ten percent (1 out of 10) of the CB2 posttest protocols.

4.3.4 Mathematical Concepts Related to Specific Conceptualizations

Statistical hypothesis testing (see section 4.3.1) suggested there was no difference in mathematical concept posttest scores for participants that received fair-share instruction compared to participants that received center-of-balance instruction. A qualitative examination of the coded data revealed each mathematical concept problem, MC1 and MC2, was primarily solved using a particular conceptualization. That is, solutions for MC1 tended to utilize the center-of-balance conceptualization and solutions for MC2 tended to utilize the fair-share conceptualization. Table 4-7 denotes the posttest solution methods for participants in the fair-share and center-of-balance groups combined.

Table 4-7: Posttest Solution Methods for Mathematical Concept Problems

	Method of Solution			
Problem	**Fair-Share**	**Center-of-Balance**	**Alternative**[a]	**Undetermined**[b]
MC1	0	7	7	6
MC2	12	1	5	2

Note: No problem had evidence of both conceptualizations
[a]Primarily inductive argument for MC1 and primarily algebraic solution using arithmetic mean formula for MC2.
[b]Includes unsubstantiated solutions and unsolved problems

The evidence suggests that mathematical concepts of the arithmetic mean, at least the two offered by problems MC1 and MC2, may be cognitively connected to a specific conceptualization of the arithmetic mean (i.e. fair-share or center-of-balance). Participants who gained fair-share knowledge were unable to adapt it to problem MC1, but readily used the new knowledge in problem MC2. Conversely, participants who gained center-of-balance knowledge readily adapted it to problem MC1, but in only one case used it to solve problem MC2.

5.0 DISCUSSION

The relatively simple calculation for quantifying the arithmetic mean can obscure its connection to other knowledge spaces that help cultivate its understanding. Two such knowledge spaces, the notions of fair-share and center-of-balance, were the focus of this investigation. Research indicates most students view the arithmetic mean as a procedure (McGatha, Cobb, & McClain, 2002), and often do not understand it as a fair-share distribution of the data (Mokros & Russell, 1995) or as the center-of-balance of the data set (Hardiman et al., 1984). Furthermore, articulating a connection between the conceptualizations of fair-share and center-of-balance is often difficult even for those with advanced understanding in statistics (MacCullough, 2007). Linking the two conceptualizations of the arithmetic mean with each other and with mathematical concepts connects these fragments of knowledge and thus constructs a web of understanding for the arithmetic mean. The purpose of this study was to explore how liberal arts university students connect the different conceptualizations of the arithmetic mean (i.e. fair-share and center-of-balance) to one another and to related mathematical concepts.

This discussion is presented in three parts. First, the results of the study are explained and situated within the existing literature base. Next, implications of the findings and recommendations, particularly as they relate to pedagogy, are discussed. Finally, directions for future reach are offered.

5.1 EXPLANATION AND SITUATING OF RESULTS

The results of this study show that increased knowledge of the fair-share conceptualization of the arithmetic mean improved knowledge of the center-of-balance conceptualization as ascertained via problem solving. Similarly, increased knowledge of the center-of-balance conceptualization of the arithmetic mean improved knowledge of the fair-share conceptualization. Also, increased knowledge of either the fair-share or center-of-balance conceptualization advanced understanding of the mathematical concepts associated with the arithmetic mean. However, increased knowledge of each conceptualization distinctively affected specific mathematical concepts.

5.1.1 Initial Knowledge and Use of the Conceptualizations

Prior research has indicated students' primary solution strategies for arithmetic mean problems are based on the arithmetic mean formula (Cai, 1998; Groth, 2005; Groth & Bergner, 2006; Mokros & Russell, 1995). The current study confirmed this prior result. An examination of participants' written solutions on the pretest revealed a preponderant use of the arithmetic mean formula even in cases where its use was inappropriate or unfounded. For example, several participants attempted to solve problems FS2 and CB2 by means of the arithmetic mean formula. The resulting insufficient systems of equations, $\frac{3.2+3.4+x+3.1+3.2+y+3.3+3.4+3.0+3.1}{10}=3.2$ for FS2 and $\frac{\$1.30+\$1.35+t+u+v+w+x+y+z}{9}=\$1.38$ for CB2, could not be solved for the missing variables. Use of the arithmetic mean formula in problems that culminated in a correct solution or a mathematically sound solution attempt did not reveal what conceptual knowledge participants had regarding the arithmetic mean. However, similar to results of prior research (e.g. Cai, 1998; Groth & Bergner, 2006; McGatha, Cobb, & McClain, 2002; Mokros & Russell, 1995; Pollatsek, Lima & Well, 1981), evidence indicated many participants did not have methods of solution based on a conceptual understanding of the arithmetic mean for problems that were not suitably solved with the arithmetic mean formula or for situations where participants were unable to utilize the formula to construct a solution. In particular, the conceptualization of center-of-balance as it relates to the arithmetic mean seemed to be absent or minimally connected for most participants. This is not surprising; the relationship between the center-of-balance conceptualization and the arithmetic mean formula is rooted in the physical concept of center-of-mass; a more difficult and, most likely, unfamiliar concept to the participants in this study. In contrast, the conceptualization of fair-share appeared to be somewhat developed, at least in the sense of how it related to the arithmetic mean formula, for most participants even prior to instruction. The protocols showed participants were able to relate the concept of partitive division to the arithmetic mean formula, or algebraically manipulate the arithmetic mean formula to calculate the total amount for the entire data set and share it equally amongst the data points.

Participants were more successful solving fair-share problems than they were solving center-of-balance problems. This difference can be explained by combining the results from this study and from previous research with the knowledge structure proposed in section 2.4. The knowledge structure linked the mathematical domain and the statistical domain of the arithmetic mean to the conceptualizations of fair-share and center-of-balance as cognitive blending spaces. The relative strength of the connections between domains and conceptualizations may account for the difference in difficulty of fair-share and center-of-balance problems.

Evidence in the protocols indicates a strong relationship between the fair-share conceptualization and the mathematical domain. This was evident through the successful use of partitive division to represent the total accumulation being equally shared among all data points or through the successful use of the normalized ratio (Cortina, 2002) to represent the arithmetic mean is a suitable surrogate for each value in the data set in problems FS1 and FS2. The success most participants accomplished on problem MC2, a mathematical concept problem most often solved using some sense of fair-share, is further evidence of a strong relationship between the fair-share conceptualization and the mathematical domain.

The link between the center-of-balance conceptualization and the mathematical domain is, in contrast, a weak relationship within the proposed knowledge structure. The weak relationship was indicated in the protocols by the low scores on problem CB1, the minimal use of center-of-balance knowledge on problem CB2, and the low scores on problem MC1, a mathematical concept problem linked to center-of-balance knowledge. The general concept of center-of-balance, particularly the mathematical concepts relating it to center-of-mass, is only partially understood even at adulthood (Hardiman, Pollatsek, & Well, 1986; Jackson, 1965; Lovell, 1961; Siegler, 1976). Therefore, using it to model a different concept, such as the arithmetic mean, proves to be challenging. "Using one poorly understood set of ideas—the physical relationship of weights and distance—may not help students understand another set of difficult ideas—the numerical relationship between the mean and the data" (Russell & Mokros, 1996, p. 361).

Prior research (e.g. Cai, 1998; Groth & Bergner, 2006; McGatha, Cobb, & McClain, 2002; Mokros & Russell, 1995; Pollatsek, Lima & Well, 1981), as well as the results of this study, indicate the overall knowledge of the statistical domain, or the idea that the arithmetic mean is representative, is weak among most students. It is therefore not surprising to find the connection between the statistical domain and the conceptualizations of fair-share and center-of-balance are not soundly developed in most participants. However, a few participants used the conceptualizations of fair-share and/or center-of-balance to describe the arithmetic mean as representative of the data set. In particular, those participants who made use of the center-of-balance conceptualization most often related it to the statistical domain.

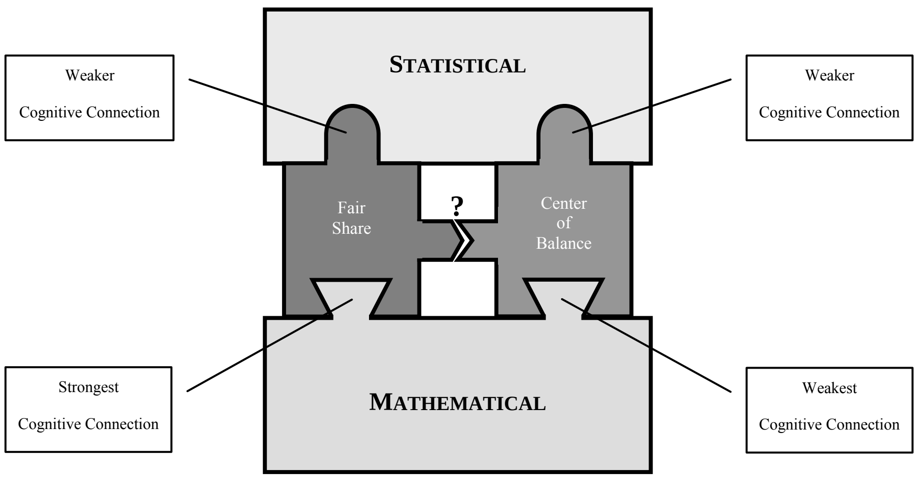


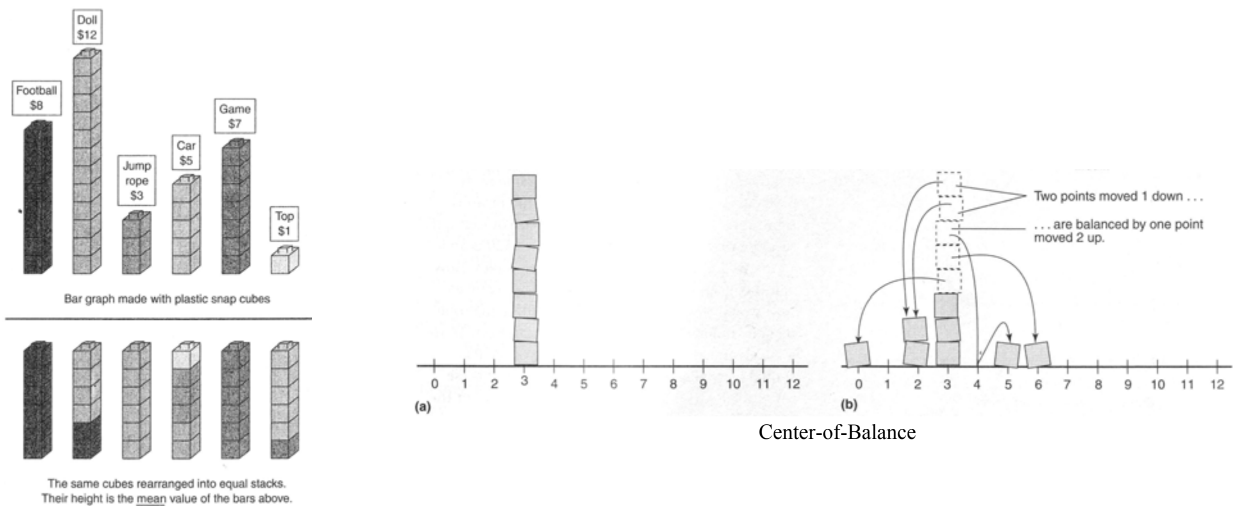
Figure 5-1: Recap of Knowledge Structure of the Arithmetic Mean

To summarize, most participants summoned knowledge of the connection between the fair-share conceptualization and the mathematical domain, namely partitive division, and used it to successfully solve problems. The reasons the participants were more successful in solving fair-share problems as opposed to center-of-balance problems were twofold. First, the only strong connection identified within the structure, Figure 5-1, was between the fair-share conceptualization and the mathematical domain. Thus, participants were better equipped with the knowledge and mathematical tools to solve fair-share related problems. Second, as discussed in section 2.3, the intricacies and nature of the blending spaces themselves are inherently different. The general concept of fair-share is understood at a very early age and its mathematical model, partitive division, is relatively simple. On the other hand, the mathematical model relating center-of-balance to the arithmetic mean (i.e. torque and center-of-mass) is comparatively difficult.

5.1.2 Connecting the Conceptualizations

There are two important reasons for cognitively connecting the fair-share and center-of-balance conceptualizations of the arithmetic mean. First, an important component of understanding the arithmetic mean is justifying in one's mind how two seemingly different conceptualizations (i.e. fair-share and center-of-balance) can describe the same concept. It is difficult to fathom a connection between the notions of fair-share and center-of-balance in the general knowledge schema outside the context of the arithmetic mean. Finding harmony between the conceptualizations may help solidify understanding of the arithmetic mean. Second, as previously discussed, access to knowledge of center-of-balance as it relates to the arithmetic mean is impeded on two fronts: (a) its mathematical context is rooted in the difficult concepts of torque and center-of-mass, and (b) the inadequate understanding students have of the statistical context of the arithmetic mean. Cognitively connecting the fair-share conceptualization to the center-of-balance conceptualization within the context of the arithmetic mean may provide an alternate and more explicit path to the concept of balance.

MacCullough (2007) found experts thought of the connection between fair-share and center-of-balance as a *leveling-off* conception. Figure 5-2 depicts how the experts "visualized" the process for both the fair-share and center-of-balance conceptualizations.



Fair-Share

Center-of-Balance

(Van de Walle & Lovin, 2006)

Figure 5-2: Fair-Share and Center-of-Balance as Leveling-off

"They [the experts] moved seamlessly between the two conceptions [fair-share and center-of-balance] using the conception with which it was easiest to work in the given task." (MacCullough, 2007, p. 100). The participants in the current study used the notion 'the sum of the deviations from the mean is zero' to transfer knowledge from one conceptualization to the next. The two relationships (i.e. leveling-off and sum of the deviations from the mean is zero) essentially model the same concept, but they define the connection between fair-share and center-of-balance from two different domains (i.e. statistical and mathematical). Experts view the mean as a statistically representative quantity. The experts could visualize how the leveling-off process both equally distributes and balances the data. In contrast, the majority of participants in the current study lacked knowledge of the statistically representative notion and had only a mathematical perspective of the arithmetic mean. Therefore, the connection between fair-share and center-of-balance manifest as one of Strauss and Bichler's (1988) mathematical properties of the mean, the sum of the deviations from the mean is zero.

5.1.2.1 Hypothesized Connection between the Conceptualizations

It is not surprising that the participants focused on the concept, 'the sum of the deviations from the mean is zero,' in the posttest solutions as it was an integral part of the instruction for both the fair-share and center-of-balance group. What is noteworthy, however, is the transfer of newly learned knowledge into problems that centered on a different conceptualization. Participants seldom used the explicit fair-share or center-of-balance knowledge that they learned in the instructional modules on posttest problems of the other conceptualization. That is, participants did not employ a balance model learned in the center-of-balance module to solve a fair-share posttest problem; nor did they redistribute the original data set as learned in the fair-share module to solve a center-of-balance posttest problem. Rather, they took a concept learned in the context of each conceptualization, 'the sum of the deviations from the arithmetic mean is zero,' and transferred it to the other conceptualization. The following is a hypothesis of how participants may have transferred the 'sum of the deviations from the mean is zero' concept between the fair-share and center-of-balance conceptualizations.

Within a knowledge structure, the existence and strength of connections between concepts and/or schema have limitations. One factor that determines how vast the connections are is the generality in which a concept is learned (J. Greeno personal communication, July 15, 2008). In the case of this study, learning from a specific instruction module (i.e. fair-share or center-of-balance) had to be general enough to transfer into the other conceptualization. The concept, 'the sum of the deviations from the mean is zero,' was a general idea learned in the more specific context of either fair-share or center-of-balance. This concept is not limited to connections to only one conceptualization; rather, it can be generalized to both. In other words, the form of the schema for the concept did not require all the qualities of a specific conceptualization to activate use of the concept. The protocols indicated that some participants certainly related the center-of-balance conceptualization to equal deviations from the mean. They were able to equally "balance" the differences above and below the mean. The protocols also indicated some participants related the fair-share conceptualization to deviations from the mean. These participants saw the arithmetic mean as having a property of the data or distribution that shared the "extra" data equally. That is, data points not equal to the mean had to "give" or "receive" data such that all data was shared equally. The ability to generalize the 'sum of the deviations from the mean is zero' concept afforded participants the opportunity to transfer the knowledge between the fair-share and center-of-balance conceptualizations.

5.1.3 Importance of Both Conceptualizations

While results from research question #1 indicated certain knowledge (e.g. sum of the deviations from the mean is zero) can be transferred from one conceptualization (i.e. fair-share or center-of-balance) to the other; results from research question #2 indicated that knowledge of only one conceptualization may not be sufficient to solve all arithmetic mean problems, and therefore does not offer a complete understanding of the arithmetic mean. A holistic understanding of the arithmetic mean includes relying on both blending spaces (i.e. fair-share and center-of-balance) to connect the statistical and mathematical knowledge domains.

The mathematical concept problems, MC1 and MC2, were each almost exclusively solved using one of the conceptualizations, center-of-balance or fair-share, respectively. The mathematical concept in problem MC1 characterized two of Strauss and Bichler's (1988) mathematical properties of the arithmetic mean: (a) when one calculates the average, a value of zero, if it appears, must be taken into account, and (b) the average is influenced by values other than the average. One of two general solution strategies stood out in the majority of protocols: (a) an inductive method based solely on the arithmetic mean formula, or (b) a depiction of the arithmetic mean as the center-of-balance. The fair-share conceptualization of the arithmetic mean was not perceptible in any of the posttest protocols for problem MC1. Although it is possible to illustrate the two mathematical properties relevant to problem MC1 as a fair-share conceptualization; participants in this study considered these properties applicable to the center-of-balance conceptualization regardless of the focus of their instruction (i.e. fair-share or center-of-balance).

The mathematical nature of problem MC2 corresponded to Cortina's (2002) view of the arithmetic mean as a normalized ratio. In this case, the arithmetic mean is described as an attribute of a group of data points in which an aggregate measure is created by summing all of the individual data point values. This notion, although plausibly depicted as center-of-balance, is predominately a fair-share perspective of the arithmetic mean. Consequently, participants in this study, except in one case, used a fair-share conceptualization to conceive of a solution to the problem.

The results from the mathematical concept problems indicate there are properties and attributes of the arithmetic mean that are perceived by students as a particular conceptualization (i.e. fair-share *or* center-of-balance). Increasing knowledge of both conceptualizations may permit students access to awareness of particular properties and attributes of the arithmetic mean not readily recognized through knowledge of only one conceptualization. Results from prior research studies revealed increased knowledge of the fair-share conceptualization led to a more conceptual understanding of the arithmetic mean as representative of a data set (Cai & Moyer, 1995; George, 1995; Groth; 2005). Similar results were demonstrated in research that focused on the center-of-balance conceptualization of the arithmetic mean (Hardiman et al., 1984). These results, along with the results of the current study, indicate increased knowledge through instruction of both conceptualizations, including cognitively connecting the two conceptualizations through the concept 'the sum of the deviations from the mean is zero,' could provide a more comprehensive understanding of the arithmetic mean.

5.2 IMPLICATIONS AND RECOMMENDATIONS

It has been suggested that the ideal scaffold for learning statistical concepts such as the arithmetic mean is to first develop its statistical sense, in the case of the arithmetic mean—representativeness, and then connect this conceptual understanding to the governing mathematical aspects (Jones et al., 2004; Konold & Higgins, 2003; Mokros & Russell, 1995). MacCullough's hypothesis as to how the experts in her study acquired their knowledge of the arithmetic mean seemed to follow this learning process (see MacCullough, 2007, pp. 100-102). It is clear that many participants in the current study had a narrow perspective of the knowledge that constitutes the arithmetic mean. Their perspective was a mathematically based fair-share point of view derived from the arithmetic mean formula, not from a developed sense of the arithmetic mean as representative of the data set. It is highly likely that the participants developed their mathematical knowledge of the arithmetic mean absent from the statistical sense of representativeness. In doing so, new knowledge related to the arithmetic mean was, if possible, connected to the arithmetic mean formula, or otherwise undesirably situated as an unconnected fragment in the knowledge schema.

Curricula and instruction that only present the arithmetic mean as a formulaic procedure or do not conceptually develop the arithmetic mean using both conceptualizations (i.e. fair-share and center-of-balance) may lack opportunities for students to make relevant cognitive connections between the conceptualizations, and between the conceptualizations and mathematical concepts. Huberty, Dresden, and Bak (1993) in their study on the dimensions of statistical knowledge suggest:

> Students have a relatively poor grasp of the conceptual understanding of statistics, it is especially recommended that instructors encourage students to think in terms of multiple ideas and connections among them (i.e., to develop conceptual understandings from their studies). Making connections between ideas and skills may provide the foundation for richer understanding and greater ability to make use of statistical methods in the future (p. 531).

Providing curricula and instruction that encourage the use of both the fair-share and center-of-balance conceptualizations as blending spaces provides the opportunity for students to think of the arithmetic mean in terms of multiple ideas. Connecting mathematical concepts through one or both conceptualizations provides the blending spaces necessary to relate the statistically conceptual ideas to the mathematical concepts and skills that constitute the arithmetic mean; thus providing the foundation for richer understanding.

Several reform mathematics curricula construct a conceptual sense of the arithmetic mean. Curricula such as *Connected Mathematics* and *Investigations in Number, Data, and Space*, for example, offer students the opportunity to build a representative sense of the arithmetic mean as a statistical element by working with and manipulating data sets. Previous researchers investigating the development of knowledge in statistics (Jones et al., 2000; Mooney, 2002), and in particular the arithmetic mean (Konold & Higgins, 2003; Mokros & Russell, 1995), have concluded that the conceptual underpinnings of statistical ideas like the arithmetic mean need to be developed before the procedures for their calculations are introduced. Without conceptual underpinnings, the formula to calculate the arithmetic mean becomes the prevailing sense of average for many students and grows only in procedural complexity (Groth & Bergner, 2006; Leon & Zawojewski, 1990; Watson & Moritz, 2000). The predominance of the participants in this study to use the arithmetic mean formula, particularly when it was used ineffectively or inappropriately, may indicate that they learned the arithmetic mean as a procedural formula with little, if any, conceptual basis. Garfield (1995) asserted that "students' misconceptions are often strong and resilient—they are slow to change even when students are confronted with evidence that their beliefs are incorrect" (p. 32). Evidence from this study verified the statement and, in fact, it showed increased conceptual knowledge of the arithmetic mean does little to affect the procedural solution strategies of students without misconceptions. First, a few students who incorrectly solved problems using the arithmetic mean formula on the pretest used the same incorrect or incomplete method on the posttest regardless of the instruction. Second, there were only three cases in which students who correctly used an alternative method of solution (e.g. the arithmetic mean formula) on the pretest changed their method of solution to one that utilized either the fair-share or center-of balance conceptualization on the posttest. Participants' connection to the formula proved to be incredibly strong for both those that used it inappropriately and for those that used it effectively.

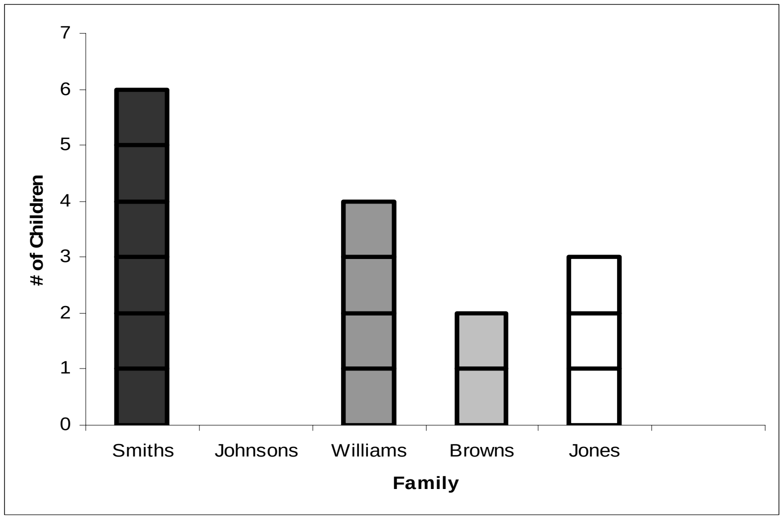
Instructors, therefore, have a difficult task when reliance on the arithmetic mean formula takes root before conceptual understanding of the arithmetic mean is established. Establishing or advancing the statistical sense (i.e. representativeness) of the arithmetic mean may be inhibited by the procedural nature of the arithmetic mean formula. Instruction focusing on 'the sum of the deviations from the mean is zero' may be the key to linking the mathematical and statistical aspects along with the fair-share and center-of-balance conceptualizations of the arithmetic mean.

5.2.1 Using the 'Sum of the Deviations from the Mean is Zero'

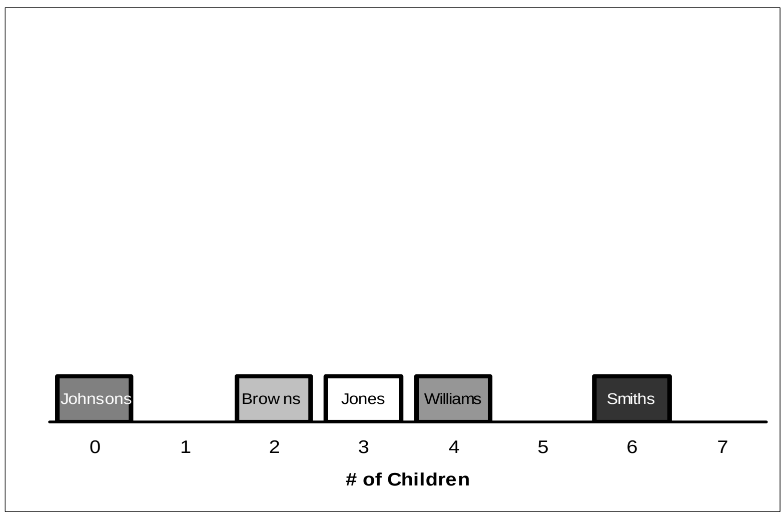
The findings of this research study indicate the importance of one of Strauss and Bichler's (1988) fundamental properties of the arithmetic mean, 'the sum of the deviations from the mean is zero.' The instruction modules utilized in this study did not explicitly relate the fair-share and center-of-balance conceptualizations; yet, participants were able to use the property to solve problems that focused on a different conceptualization from which their instruction had initially connected the property. One might expect then, that instruction focused on both conceptualizations, particularly a connection between the conceptualizations (e.g. the sum of the deviations from the mean is zero), would have an even greater impact on students' conceptual understanding of the arithmetic mean. Approaches to instruction that contain this focus are described in this section.

As previously noted, MacCullough, (2007) found experts used the leveling-off strategy to connect the conceptualizations of fair-share and center-of-balance. The leveling-off idea can be used as a visual model to relate the 'sum of the deviations from the mean is zero' in both conceptualizations. Van de Walle and Lovin (2006) recommend the following exercise to connect the models of the two conceptualizations.

Give the students (or have them create) two different graphical representations of the same data (i.e. (a) bar graph for fair-share and (b) frequency distribution for center-of-balance) as depicted in Figure 5-3. Instruct the students to level the bars in (a) by only moving one cube at a time from a longer bar to a shorter bar. Each time they move a cube off of a bar in (a), the cube denoting that bar in (b) must be moved one deviation to the left. At the same time, the cube in (b) denoting the bar on which the cube in (a) was added must be moved one deviation to the right. The movements illustrate how deviations from the mean are related in both the fair-share and center-of-balance models. The exercise also emphasizes the notion that the arithmetic mean is representative of the data set, not just a mathematical calculation. The instruction relates the mathematical concept, 'the sum of the deviations from the mean is zero,' valued by the sample population of this study, to the leveling-off visualization that experts used to equate the two conceptualizations.



(a)



(b)

Figure 5-3: Graphical Representations of Conceptualizations

A second instructional plan that would construct a connection between the fair-share and center-of-balance conceptualizations was offered by J. Greeno (personal communication, July 15, 2008). Similar to the instruction modules used in this study, initial instruction would center on one conceptualization guided by the concept, 'the sum of the deviations from the mean is zero.' A preliminary assessment, consisting of problems similar to the fair-share and center-of-balance problems in this study, would be given that offered problems related to both conceptualizations. Results from this study indicate at least some students would transfer knowledge from the initial instruction to the problems on the assessment related to the other conceptualization. An orchestrated class discussion of the assessment would draw upon the solutions of students who transferred knowledge between the conceptualizations. The discussion would exemplify the connection between the conceptualizations and construct the relationship between fair-share and center-of-balance.

These examples of instruction, or instruction similar to it, link the fair-share and center-of-balance conceptualizations. The generalized concept, 'the sum of the deviations from the mean is zero,' which permeates both conceptualizations, is used as a cognitive bridge between them. The instruction emphasizes knowledge of the arithmetic mean from the perspective of both conceptualizations; thus expanding the blending space for students to connect the mathematical and statistical domains.

5.3 CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

The results of this study are significant for several reasons. First, participants in this study used the concept 'the sum of the deviations from the mean is zero' to connect the fair-share and center-of-balance conceptualizations of the arithmetic mean. The concept was transferred bi-directionally; that is, the concept was transferred in an almost equal number of cases from fair-share to center-of-balance and from center-of-balance to fair-share. Neither conceptualization seemed to be more effective than the other in promoting transfer of knowledge. The focus of this study was not on instruction, but the results suggest the cognitive link between the conceptualizations may be pedagogically significant. Future research should examine the extent to which instruction linking both conceptualizations improves students' understanding of the arithmetic mean. In particular, does this type of instruction lead to a better understanding of the arithmetic mean as statistically representative of a data set?

Second, participants' protocols indicated knowledge of each conceptualization was important in solving problems related to specific mathematical properties of the arithmetic mean. Therefore, knowledge of only one conceptualization, either fair-share or center-of-balance, does not sufficiently portray all aspects of the arithmetic mean. The weight each conceptualization merits during instruction might be an area of future investigation. Given the restricted time frame in most curricula; is there an optimum distribution of resources spent individually characterizing and/or collectively relating the conceptualizations?

Third, the arithmetic mean has uses in statistics beyond the suggestion of central tendency. Future research could relate how knowledge of a particular conceptualization of the arithmetic mean, (i.e. fair-share or center-of-balance) affects knowledge of other concepts in statistics, such as variance and distributions.

Fourth, this study proposed a knowledge structure for the arithmetic mean. Within the structure, the conceptualizations of fair-share and center-of-balance acted as blending-spaces combining ideas cultivated in the mathematical and statistical domains that compose the knowledge of the arithmetic mean. Future studies should refine the knowledge structure and further define where particular concepts related to the arithmetic mean fit and interact within the structure.

In summary, this study has provided data as to how the fair-share and center-of-balance conceptualizations interact with each other and inform mathematical concepts in the context of the arithmetic mean. In a theoretical sense, the interactions cognitively connect the blending-spaces proposed in the knowledge structure for the arithmetic mean. In a practical sense, these interactions may signify strategies for improving student understanding of the arithmetic mean by focusing instruction on the concept, 'the sum of the deviation from the mean is zero,' to connect the fair-share and center-of-balance conceptualizations.

APPENDIX A

PROBLEMS FROM PILOTED INSTRUMENTS NOT INCLUDED IN STUDY

Attempted Measure	Source	Statement of Problem	Reason for Exclusion
Fair-Share	(Strauss & Bichler, 1988)	For a class party, Ruth brought 5 pieces of candy, Yael brought 10 pieces of candy, Nadav brought 20 pieces of candy, and Ami brought 25. Can you tell me in one number how many pieces of candy each child brought? How did you decide on that number?	The solution too often used only the arithmetic mean formula without evidence of conceptual understanding or underlying knowledge.
Fair-Share	(Strauss & Bichler, 1988)	We took some numbers and added them up. Before we added them, the largest number we had was 5. Afterwards, we divide up the added numbers equally, and we ended up with six. Do you think this could happen?	The problem was often misconceived or too often answered without written artifact or complete verbalization of thoughts.
Fair-Share	Expert in statistics education	Three children went on an Easter egg hunt. John found 9 eggs, Betty found 5 eggs, and Ty found 4 eggs. If the total eggs were going to be divided so each child received an equal number, how many eggs would each child get?	The problem was too often solved using only the arithmetic mean formula without evidence of conceptual understanding or underlying knowledge.

continued

Attempted Measure	Source	Statement of Problem	Problem Analysis
Center of Balance	(Aufmann, Lockwood, Nation, & Clegg, 2007) Author	If one number in a data set is changed, will it necessarily change the mean of the set? Explain. If two numbers in a data set are changed, will it necessarily change the mean of the set? Explain.	The first part of the problem was too often answered “yes” or “no” without explanation or, if prompted, ability to explain. The second part of the problem seemed to encourage a quick response without thorough consideration.
Center of Balance	(Cai, Moyer, & Grochowski, 1999).	We took a survey of the family size of ten different families. The average (mean) family size for these ten families was 4. What could the family sizes of each of these ten families be?	The problem was too often answered without written artifact or complete verbalization of thoughts.
Center of Balance	(Freedman, Pisani, & Purves, 1998) Author	Ten people in a room have a mean height of 5 feet 6 inches. An 11^{th} person enters the room, what height would they be if the mean height was now 5 feet 7 inches? Ten people in a class have a mean height of 5 feet 6 inches. An 11^{th} person enters the room, what height would they be if the mean height of all people remains at 5 feet 6 inches?	The first part of the problem encouraged use of the arithmetic mean formula. This seems to have influenced the plan of solution for the second part of the problem as most solvers attempted to use the formula without revealing conceptual understanding or accessed knowledge.

continued

Attempted Measure	Source	Statement of Problem	Problem Analysis
Mathematical Concept	(MacCullough, 2007).	My sister and I went for a drive and decided we would share the driving time equally. For the first 4 hours of the journey my sister drove 70mph. I drove a bit slower for the second four hours and averaged 50mph. What was the average speed for the entire trip? Would this be different if we drove 70mph and 50mph respectfully but instead of splitting the driving by hours (time), we divided the trip into two halves by miles (distance)?	The knowledge evidenced in solving the problem showed no relationship to the conceptualizations of fair-share or center-of-balance and therefore transfer of knowledge from those statistical concepts to the mathematical concepts of this problem was unlikely at best.
Mathematical Concept	(Strauss & Bichler, 1988)	Children brought cookies to a party they were having. Some children brought many and some brought few. The children who brought many gave some to those who brought few until everyone had the same number of cookies. Was the number of cookies given by those who brought many more than, the same as, or less than the number of cookies received by those who brought few? Why?	The problem was often misunderstood and therefore solutions were ill-conceived or did not relate to the concepts of fair-share or center-of-balance.
Mathematical Concept	(Mevarech, 1983)	The mean number of units produced by 100 workers at factory A is 52.6. The mean number of units produced by 50 workers at factory B is 31.8. The two factories merged together, what is the mean number of units produced by all workers in the merged factory?	The knowledge evidenced in solving the problem showed no relationship to the conceptualizations of fair-share or center-of-balance and therefore transfer of knowledge from those statistical concepts to the mathematical concepts of this problem was unlikely at best.

APPENDIX B

PROTOCOL ANALYSIS TRAINING SESSION

B.1 INSTRUCTIONS

In this experiment I am interested in what you think when you solve problems related to the arithmetic mean. In order to do this I am going to ask you to think aloud as you write solutions to problems given. What I mean by think-aloud is that I want you to tell me everything you are thinking from the time you first see the problem until you finish solving it. I would like you to talk aloud constantly from the time I present you each problem until you have given your final answer. I do not want you to try to plan out what you are going to say or try to explain to me what you are saying. Just act as if you are alone in the room speaking to yourself. It is most important that you keep talking. If you are silent for any long period of time I will ask you to talk. Do you understand what I want you to do?

(Ericson & Simon, 1993)

The following problems are about the arithmetic mean. Each of them can be solved correctly and efficiently using several different approaches or techniques. You can use any method or combination of techniques you wish as you solve each of the problems.

(Ericson & Simon, 1993)

B.2 WARM-UP EXERCISES

Multiply these two numbers using your paper and pencil and tell me what you are thinking as your work toward the answer.

24 x 36 =

Think aloud, telling me everything you are thinking as you solve this problem. You may use your pencil and paper if you wish:

How many windows are in your parent's house?

In solving this task you should think aloud. If I remind you to do so during the process please immediately verbalize what you are thinking.

Generate as many words as possible using the letters "ONDTERH"

(Ericson & Simon, 1993)

APPENDIX C

KNOWLEDGE INSTRUCTION MODULES

C.1 FAIR-SHARE AND RELATED MATHEMATICAL CONCEPTS

Fair-Share Knowledge Intervention

Objective: Students will be able to:

1. utilize knowledge of fair-share to solve arithmetic mean problems
2. relate knowledge of fair-share to mathematical concepts of the arithmetic mean.

Materials:

Access code to Blackboard online learning system

Introduction:

Participants asked to write down their own definition of the arithmetic mean.
Participants are asked to compare their definition of the mean with the following description of the mean:

> The arithmetic mean is a quantity (number) that is statistically representative of an entire data set. Conceptually this quantity can be thought of as a fair-share distribution or center-of-balance of the data set.

Participants will be informed they are going to study the mean as a fair-share distribution.

Procedure:

Participants will complete the interactive lesson including notes, problems, solutions, and video. (Annenberg Media, 2002)

Part A:
Fair Allocations

The term average is a popular one; it is often used, and often used incorrectly.

Although there are different types of averages, the typical definition of the word "average" when talking about a list of numbers is "what you get when you add all the numbers and then divide by how many numbers you have." This statement describes how you calculate the arithmetic mean, or average. But knowing how to calculate a mean doesn't necessarily tell you what it represents.

Let's begin our exploration of the mean: Using your 45 coins, create 9 stacks of several sizes. You must use all 45 coins, and at least 1 coin must be in each of the 9 stacks. It's fine to have the same number of coins in multiple stacks.

Here is one possible arrangement, or allocation, of the 45 coins:

Problem A1 SOLUTION

Record the number of coins in each of your 9 stacks. What is the mean number of coins in the 9 stacks?

Problem A2 SOLUTION

Create a second allocation of the 45 coins into 9 stacks.

a. Record the number of coins in each of your 9 stacks, and determine the mean for this new allocation.

b. Why is the mean of this allocation equal to the mean of the first allocation?

c. Describe two things that you could do to this allocation that would change the mean number of coins in the stacks.

Problem A3 SOLUTION

Create a third allocation of the 45 coins into 9 stacks in a special way:

- First take a coin from the pile of 45 and put it in the first stack.
- Then take another coin from the pile and put it in the second stack.
- Continue in this way until you have 9 stacks with 1 coin each, and 36 coins remaining.
- Take a coin from the pile of 36 and put it in the first stack.
- Then take another coin from the pile and put it in the second stack.
- Continue in this fashion until all of the remaining coins have been used.

a. Now record the number of coins in each of your 9 stacks, and determine the mean for this new allocation.

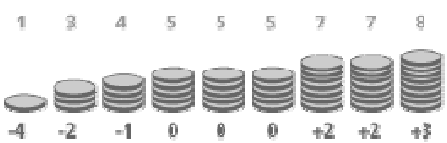
b. What observations can you make about the mean in this special allocation?

This method produces what is called a fair or equal-share allocation. Each stack, in fact, contains the average number of coins. You might think of this as a fair allocation of the 45 coins among 9 people: Each person gets the same number of coins.

Video Segment
In this video segment, Professor Kader asks participants to create snap-cube representations of the number of people in their families. He then asks them to find a way of finding the mean without using calculation. Watch this segment for an exploration of the mean.

How does the mean relate to the fair allocation of the data?

Part B:
Deviations from the Mean



Remember that the total of the excesses above the mean must equal the total of the deficits below the mean. In this case, each adds up to 7.

If you denote the values of excesses as positive numbers and deficits as negative numbers, then the total of the excesses is:

(+2) + (+2) + (+3) = +7

The total of the deficits is:

(-4) + (-2) + (-1) = -7

Statisticians refer to these excesses and deficits as deviations from the mean. For this allocation, the deviations from the mean are recorded in the table below.

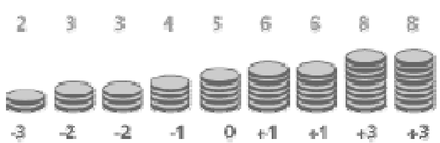
Note that the deviations always sum to 0 because the total excesses (positive deviations) must be the same as the total deficits (negative deviations).

Number of Coins in Stack	Deviation from the Mean
1	-4
3	-2
4	-1
5	0
5	0
5	0
7	+2
7	+2
8	+3
+ ______	______
45	0

Problem D1

SOLUTION

Here is another allocation of our 45 coins divided into 9 stacks:



b. Complete the following table of deviations:

Number of Coins in Stack	Deviation from the Mean
2	-3
3	
3	
4	
5	
6	
6	
8	
8	+3
+ ______	______
45	0

SHOW ANSWERS

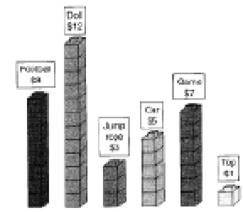
When the positive and negative deviations are added together, the total is always 0. This property illustrates another way to interpret the mean.

Participants will complete the following activities using linking blocks and/or small *Post-it* notes.

Exercise A (Van de Walle & Lovin, 2006)

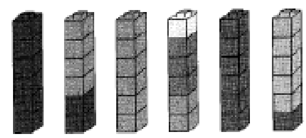
Use the linking blocks to create a bar graph representing the prices of the following toys:

Item	Price
Football	$8
Doll	$12
Jump rope	$3
Car	$5
Game	$7
Top	$1



Bar graph made with plastic snap cubes

Redistribute the *linking blocks* in the graph to represent the cost of the toys in they were all the same price assuming the total for all the toys remains the same.

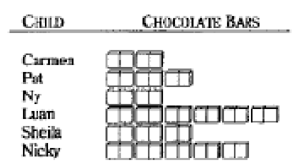


The same cubes rearranged into equal stacks.
Their height is the mean value of the bars above.

How does the number of blocks before the redistribution compare to the number of blocks after the redistribution?

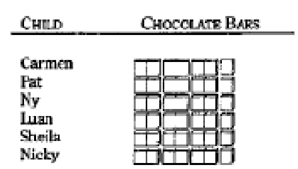
Exercise B (Uccellini, 1996)

Six children counted the number of chocolate bare that they won at the school fair. They had won 2, 3, 2, 6, 3, and 5 respectively. What is the mean number of chocolate bars they had won?
Use the *Post-it* notes to create a bar graph representing how many chocolate bars each student had won.



Redistribute the *Post-it* notes in the graph to represent the number of chocolate bars if each child would have had needed to win if they all won the same amount and the total did not change .

Hint: Use two post it notes to represent each chocolate bar.

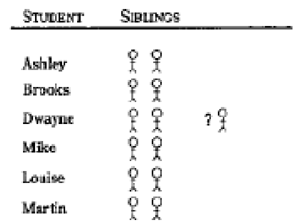


On the original graph draw a line that represents the mean

How do the number of data points above the line compare with the number of empty paces below the line?

Exercise C (Uccellini, 1996)

Six children were asked how many brother and sisters they had, and the following data were colleted form the children, 2, 1, 0, 3, 6, 1. What is the mean number of brother and sisters of these children?
Use the *Post-it* notes to create a bar graph representing how many siblings each student has.
Redistribute the *Post-it* notes in the graph to represent the number of siblings if each child would have if they all had the same number of siblings and the total did not change.



How is this problem different from the last two?
How can we resolve the issue?
What does this say about the mean?

Closure:

Redistributing data in a data set (without adding or subtracting data points of data) so that each data point has an equal allocation or fair-share of the data is equivalent to using the arithmetic mean formula to find the value of the mean.

C.2 CENTER-OF-BALANCE AND RELATED MATHEMATICAL CONCEPTS

Center-of-Balance Knowledge Intervention

Objective: Students will be able to:

1. utilize knowledge of center-of-balance to solve arithmetic mean problems
2. relate knowledge of center-of-balance to mathematical concepts of the arithmetic mean.

Materials:

Access code to Blackboard online learning system

Introduction:

Participants asked to write down their own definition of the arithmetic mean.
Participants are asked to compare their definition of the mean with the following description of the mean:

> The arithmetic mean is a quantity (number) that is statistically representative of an entire data set. Conceptually this quantity can be thought of as a fair-share distribution or center-of-balance of the data set.

Participants will be informed they are going to study the mean as a center of balance of the data set.

Procedure:

Participants will complete the interactive lesson including notes, problems, solutions, and video. (Annenberg Media, 2002)

Part A:
Line Plots

Suppose you have nine stacks of coins as shown below. If you reshuffle your stacks of coins just a bit, you can create a line plot representation that corresponds to the number of coins in each of the nine stacks, which will allow you to explore an interpretation of the mean.

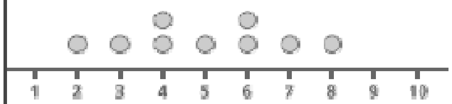
To do this yourself, create a line plot on your paper. Across the bottom of the page, draw a horizontal line with 10 vertical tick marks numbered from 1 to 10 (placed far enough apart for an adhesive dot or note to fit between each). Your number line should look like this:

Set up your 45 coins in this ordered allocation:

Now arrange the stacks of coins on the paper above the number that corresponds to the height of each stack, like this:

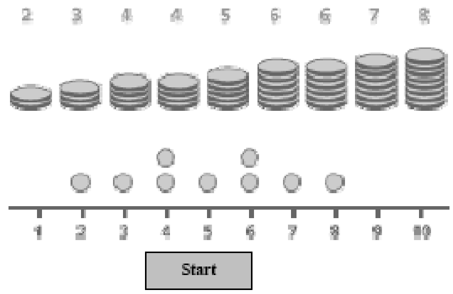
Note that the 2 stacks of size 4 and the 2 stacks of size 6 are placed above the same number.

To form your line plot, replace each of the stacks with an adhesive dot or note. You should now have a line plot that looks like this:



Each dot in the line plot corresponds to a stack of that specified size.

The following Interactive Illustration recaps the transition from physical stacks of coins to a graphical line plot representation of the stacks.



Problem C1 SOLUTION

Use the method described above to create a line plot for the following allocation of 45 coins:

Problem C2 SOLUTION

Create a line plot for this allocation of 45 coins:

Problem C3 SOLUTION

Create a line plot for this allocation of 45 coins:

PLAY!

PLAY VIDEO

Video Segment

In this video segment, participants compare their ordered stacks of snap-cubes to a line plot of the same data. Watch this segment after completing Problems C1-C3 to observe the transition from a physical representation of the data to a graphical representation.

What is one stack of snap-cubes equivalent to on the line plot of the data?

Part B:

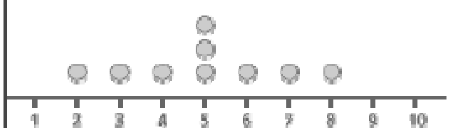
Balancing Excesses and Deficits

In the previous examples, you explored line plot representations for sets of 45 coins, each in 9 stacks. For each allocation the mean was 5 coins. Now let's use these line plot representations to explore another way to interpret the mean.

Problem C4

SOLUTION

Here is a line plot corresponding to an allocation of 45 coins in 9 stacks:



From this line plot, we can see that there are 3 stacks containing exactly 5 coins each, and 1 stack containing 6 coins. The maximum number of coins in a stack is 8, and the minimum is 2.

Rearrange the nine dots to form a line plot with each of these requirements:

a. Form a different line plot with a mean equal to 5.

b. Form a line plot with a mean equal to 5 that has exactly 2 stacks of 5 coins.

c. Form a line plot with a mean equal to 5 but a median not equal to 5.

d. Form a line plot with a mean equal to 5 that has no 5-coin stacks.

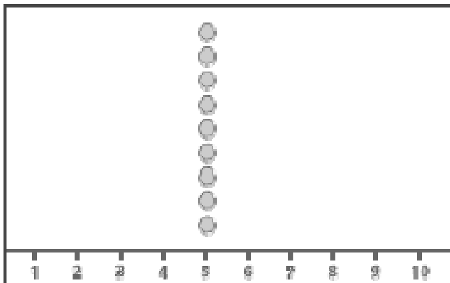
e. Form a line plot with a mean equal to 5 that has two 5-coin stacks, 4 stacks with more than 5 coins, and 3 stacks with fewer than 5 coins.

f. Form a line plot with a mean equal to 5 that has two 5-coin stacks, 5 stacks with more than 5 coins, and 2 stacks with fewer than 5 coins.

g. Form a line plot with a mean equal to 5 that has two 5-coin stacks, two 10-coin stacks, and 5 stacks with fewer than 5 coins.

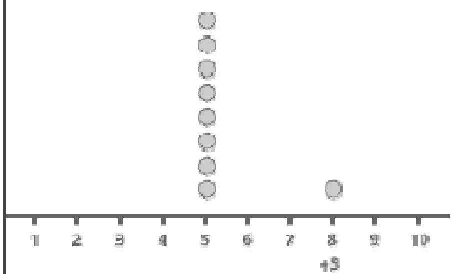
TIP Don't forget that the mean must always be equal to 5. If you move a dot to the right, it will increase the mean. Each time you move a dot to the right, you must balance this by moving another dot an equal distance to the left. Also, keep in mind that each dot represents a stack of coins, and that by moving the position of the dot, you change the number of coins in the stack. The total number of coins must remain 45.

Regardless of the strategy you used in Problem C4, you must end up with an arrangement in which the sum of the 9 values is equal to 45. Let's look at one possible strategy more closely. For the sake of simplicity, we will begin with the line plot that corresponds to the center of balance, 9 stacks of 5 coins each:



For this line plot, the sum is 45 and the mean is 5.

If we change one of the stacks of 5 coins to a stack of 8, the sum will increase by +3 to 48 and the mean will increase by +3/9 to 5 3/9. The line plot now looks like this:



Problem C5 SOLUTION

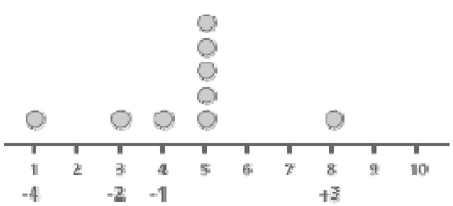
How could you change another stack of 5 coins to reset the mean to 5?

Problem C6 SOLUTION

If you could change the value of more than one stack, could you solve Problem C5 another way?

Problem C7 SOLUTION

Now suppose that we change one of the stacks of 5 to a stack of 1, which reduces the total by 4. Here is the resulting line plot:



Describe at least three different ways to return the mean to 5.

Problem C8 SOLUTION

Applying the strategy you developed in Problems C5-C7, revisit the allocations you worked with in Problem C4. You should begin with the center of balance of the 45 coins; that is, 9 dots at the mean of 5. Try to come up with answers for the questions below that are different from the ones you found in Problem C4.

a. Form a line plot with a mean equal to 5 that has exactly 2 stacks of 5 coins.

b. Form a line plot with a mean equal to 5 but a median not equal to 5.

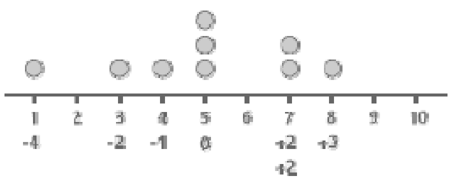
c. Form a line plot with a mean equal to 5 that has no 5-coin stacks.

d. Form a line plot with a mean equal to 5 that has two 5-coin stacks, 4 stacks with more than 5 coins, and 3 stacks with fewer than 5 coins.

e. Form a line plot with a mean equal to 5 that has two 5-coin stacks, 5 stacks with more than 5 coins, and 2 stacks with fewer than 5 coins.

f. Form a line plot with a mean equal to 5 that has two 5-coin stacks, two 10-coin stacks, and 5 stacks with fewer than 5 coins.

We are now going to explore a new way to consider excesses and deficits. Let's look at another line plot:



Problem D2 SOLUTION

Create a line plot with these deviations from the mean = 5:

(-4), (-3), (-2), (-1), (0), (+1), (+2), (+3), (+4)

Problem D3 SOLUTION

Create a line plot with these deviations from the mean = 5:

(-4), (-2), (-2), (-1), (0), (+1), (+2), (+2), (+4)

Problem D4 SOLUTION

Create a line plot with these deviations from the mean = 5, and specify a set of four remaining values:

(-4), (-3), (-3), (-1), (-1)

Problem D5 SOLUTION

How would the line plots you created in Problems D2-D4 change if you were told that the mean was 6 instead of 5?

Participants will complete the following activities using small *Post-it* notes and/or paper and pencil.

Exercise A (Van de Walle & Lovin, 2006)

Use the *Post-it* notes to create a line plot representing the prices of the following toys:

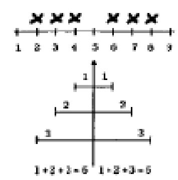
Item	Price
Football	$8
Doll	$10
Jump rope	$2
Car	$6
Game	$8
Top	$2

Move the *Post-it* notes in the plot to represent the mean cost of the toys.

How does the number of blocks before the rearranging compare to the number of blocks after the rearranging?

Exercise B (Uccellini, 1996)

Use a number line to find the mean or balancing point of the following data (3, 4, 8, 7, 2, 6).



How does this value compare to the value you would find using the arithmetic mean formula?

<u>Closure</u>:

When the positive and negative deviations are added together, the total is always 0. This property illustrates another way to interpret the mean: The mean is the balance point of the distribution when represented in a line plot, since the total deviation above the mean must equal the total deviation below the mean.

C.3 GENERAL PROBLEM SOLVING CONTROL

Chapter 1: Problem Solving and Critical Thinking

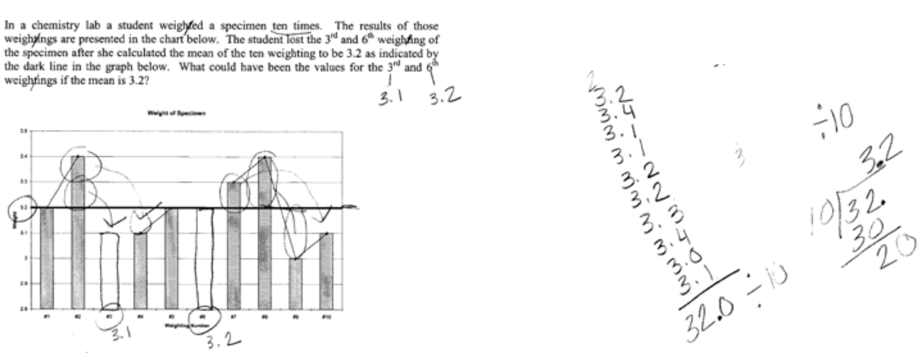
Section 3: Problem Solving

Blitzer, R. (2008). *Thinking Mathematically.* Upper Saddle River, NJ: Pearson Education Inc.

APPENDIX D

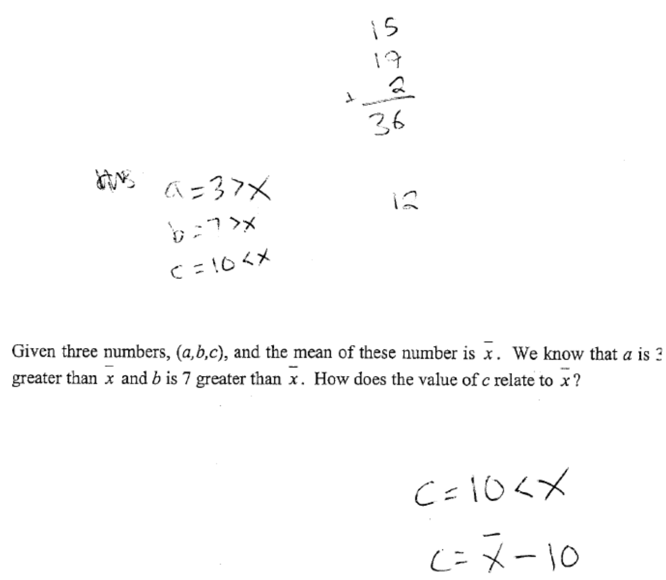
EXAMPLES OF PROTOCOLS FROM PILOT STUDY

Example #1		
Constituent Knowledge Piece	**Code**	**Explanation**
JO: Ok, so I am trying to find number three and number six. Hum, so there is one, so one, two, so I am going to have to divide by ten since there are ten.	M-FS	Counts data points and refers to arithmetic operations associated with the mean formula
JO: Or what I can do, maybe this would be easier. If I take, let's say, I take two blocks from number eight and put them down on number nine. Those two would be even. If I take one block from number seven and put it over on number ten, those two would be even. So they would all be at 3.2. If I take one block from this number two and put it down on number four, then we would only be left with one block above the line. So that means for number three, let's try this, that means only draw up to 3.1 and number six would go all the way up to 3.2; because then, for number three you take the one block above the line and bring it down onto of number three. So that means every single one would be even with the line	S-FS M	Uses the statistical conceptualization of fair-share, signified by the block-leveling strategy, the mean is represented by the blocks when the data are distributed equally to each point. The arithmetic operations of addition and subtraction aid in the block-leveling strategy.
JO: So let's see if that really works though *(talks through adding all the numbers and dividing by 10 to get 3.2).* Oh that will work. So number three is 3.1 and number six is 3.2	M	Uses the arithmetic mean formula to check answer.

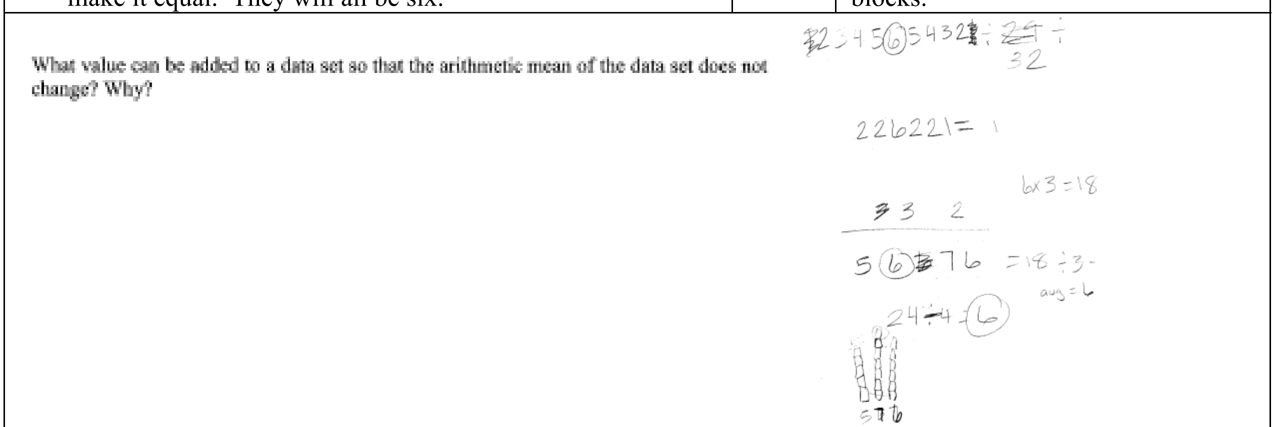


Example #2

Constituent Knowledge Piece	Code	Explanation
RK: Three greater, ok I am going to write down that *(writes a=3>x)* Ok, *b* is, *b* is seven greater *(writes b=7>x)*. How does the value of *c* relate to *x*?	M	Uses algebraic symbols to represent the mathematical relationship between the knowns and unknowns.
RK: Well, if *x* is the mean, *c* has to compensate for *a* and *b*. Umm, since *a* and *b* are greater than *x*, obviously *c* has to be lower than *x* and equal, equal part of what *a* and *b* are greater. So *a* and *b* add up to ten more than *x*. That would make c ten less than *x*.	S-CB M-CB M	Uses the notion of center-of-balance, signified by "compensate," to signify the mean is representative of the data. Uses the mathematical properties of center of balance to find the mean. Uses the arithmetic operations of addition and subtraction to employ the center-of-balance strategy.
RK: I'm going to make sure my mind is not playing tricks on me. I am going to substitute in a number. We'll say *x* is 12. Ok, so that would make me, *a* is 15, *b* would be 19 and *c* would be 2. So we have 15 plus 19 plus 12 is let's see…36. Divide by three is, let's see…12.	M	Uses algebraic substitution and arithmetic to check solution.



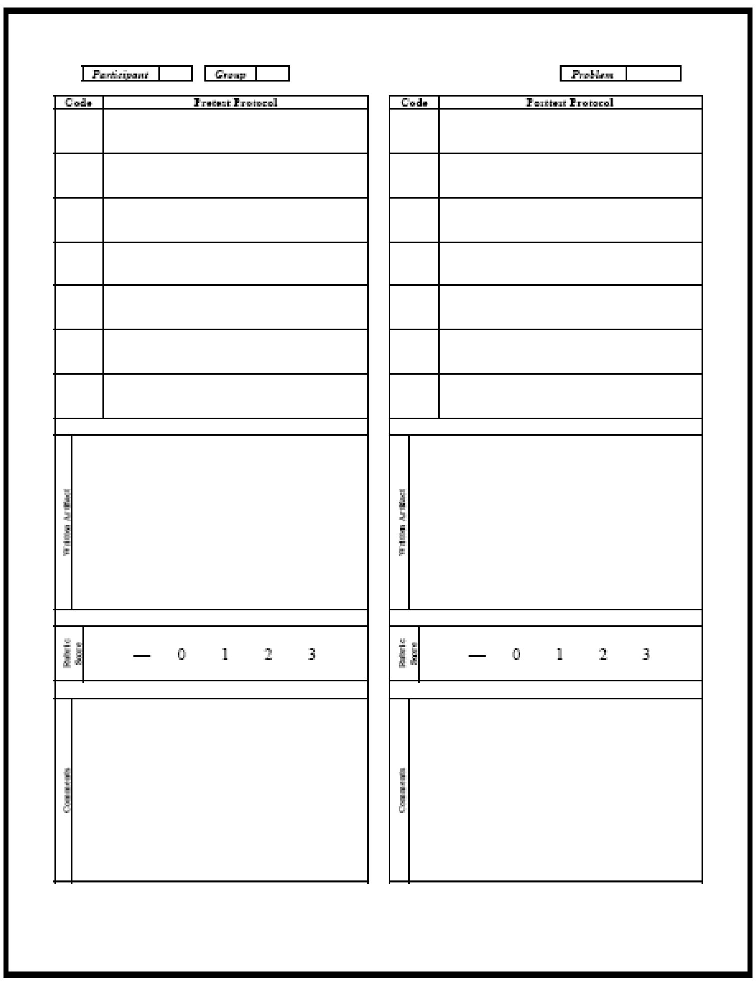
Example #3		
Constituent Knowledge Piece	**Code**	**Explanation**
JO: Well my first guess is zero but does that count as a value? Probably not.	M	Transfers the identity element of addition to the arithmetic mean.
JO: Oh I know what value can be added on, the value of the mean. So if your mean is six *(writes down 6 and circles it)* and you add that mean on again you are still going to get a mean of six. Ok, let me do one to make sure. JO: Put a five on either side *(of the circled six)*, a four on either side, a three on each side, and a two and stop there. Let's make sure the mean is six (*adds the numbers)* Twenty-nine, that does not divide evenly. Let's add a one on each side, that's better, thirty-two divided by eleven. Uh, that does not work either. JO: Let's find a number that works equally. Let's get rid of some of these numbers, too many numbers. *(counts numbers)* Nine, I don't like nine. JO: Let's do it differently. Put a six in the middle, a two here, and a two here, and a two there, and a two there. *(see 22622 representation) (pause)* MM: Keep talking JO: Ok, That is not going to work. Fourteen divided by five, let's do fifteen divided by six. No, uh, I do not want to do it that way.	S-CB M-CB M	Uses the idea of center-of-balance to represent the mean as the balancing point of the data. Misuses the mathematical sense of center of balance by placing equal number on each side of the mean rather than equal deviations. Uses arithmetic mean formula to calculate the mean of several groups of numbers.
JO: Ok, let's start an easier way. JO: Six times three is eighteen so let's do three numbers equaling six. *(writes down 2, 3, 2)* Oh, wait, they have to equal eighteen. Three number equaling eighteen, let's do six, a six, no let's do a five *(counts to figure out the seven)*, and a seven. That equals eighteen divided by three, the average equals six. So now I have my three numbers.	M-FS M	Uses the idea fair-share (partiive division) to note the total sum of three values equal to the mean.
JO: Now the mean can be added, now add another six in…twenty-four divided by four is six. So yes, you can add the mean back in so that the data set, I mean the mean does not change.	M	Adds and divides using arithmetic mean formula.
JO: Why does that work. Let me draw a picture. Draw five cubes, seven cubes, and six cubes. In order to make those even we would have to take one from the seven to the six, no, over to the five. All piles will be six. So by adding another pile of six we would not have to move any cubes to make it equal. They will all be six.	S-FS M	Begins to think about and ultimately solve the problem using fair-share to represent the mean as noted by the phrase "make those even." Adds and subtracts from piles of blocks.



APPENDIX E

EXAMPLES OF SCORING SHEET

E.1 SCORING SHEET


Participant | Group | Problem

Code	Pretest Protocol

Written Artifact

Rubric Score — 0 1 2 3

Comments

Code	Posttest Protocol

Written Artifact

Rubric Score — 0 1 2 3

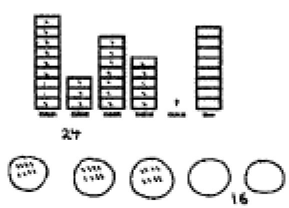
Comments

E.2 PILOT STUDY SCORING SHEETS

Participant SC **Group** CB **Problem** FS 1,3

Pretest Protocol	Code
SC: Well I know in the end everyone has to have eight and right now there are… *(pause as SC counts the blocks of Child #1, Child #2, Child #3, and Child #4)* Twenty-four - what are they - blocks, and I got five kids *(pause)* MM: Keep Talking SC: Let me draw the kids (*SC draws five circles)* One, two, three, four, five. SC: I am going to give the twenty-five, no it's twenty-four, what are they?-- blocks, to these five kids *(pause as SC draws lines to represent the blocks in each of the circles)* MM: Keep Talking SC: Let me finish this *(referring to drawing lines)* SC: I can give the first three kids there eight blocks, so the last two kids need their eight blocks. I am sixteen short so he needs to bring sixteen blocks. That is my answer.	S-FS M-FS

Written Artifact

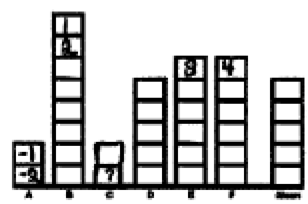


Rubric Score — 0 1 2 3

Comments

Posttest Protocol Code	Code
SC: Ok, well if the mean is in the middle I need the same amount above and below it? SC: So one, two, three, four *(counts blocks)* boxes bigger than the mean. SC: One, two below the mean. SC: Four bigger and two smaller…um, I need two more smaller. *(draws two block in C)* SC: C equals two	S-CB M-CB

Written Artifact



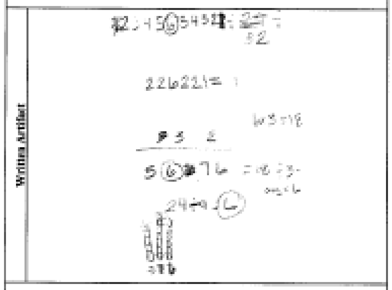
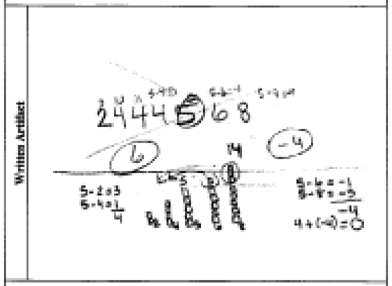
Rubric Score — 0 1 2 3

Comments

Counts blocks instead of spaces below the mean

Participant	JO	*Group*	FS	*Problem*	Pr-4,2

Pretest Protocol	Code	Posttest Protocol	Code
JO: Well my first guess is zero but does that count as a value? Probably not.	M	JO: So if the mean is five my first thought is that you are either going to get zero or you are going to get the mean.	M
JO: Oh I know what value can be added on, the value of the mean. So if your mean is six *(writes down 6 and circles it)* and you add that mean on again you are still going to get a mean of six. Ok, let me do one to make sure. JO: Put a five on either side *(of the circled six)*, a four on either side, a three on each side, and a two and stop there. Let's make sure the mean is six *(adds the numbers)* Twenty-nine, that does not divide evenly. Let's add a one on each side, that's better, thirty-two divided by eleven. Uh, that does not work either. JO: Let's find a number that works equally. Let's get rid of some of these numbers, too many numbers. *(counts numbers)* Nine, I don't like nine. JO: Let's do it differently. Put a six in the middle, a two here, and a two here, and a two there, and a two there. *(see 22622 representation)* *(pause)* MM: Keep talking JO: Ok, That is not going to work. Fourteen divided by five, let's do fifteen divided by six. No. Uh, I do not want to do it that way. JO: Ok, let's start an easier way. JO: Six times three is eighteen so let's do three numbers equaling six. *(writes down 2, 3, 2)* Oh, wait, they have to equal eighteen. Three number equaling eighteen, let's do six, a six, no let's do a five *(counts to figure out the seven)*, and a seven. That equals eighteen divided by three, the average equals six. So now I have my three numbers.	S-CB M-CB M	JO: Let me make my own data set. So if five is the mean lets put a six and say eight. So let's see that's fourteen above, so below let's put a four, a four, a four, so that's *(counts three fours)* twelve, another four, wait that's too much, we only need fourteen so how about a two. JO: Ok, so there's my data set. Let's subtract. So five minus four is one *(repeats three times)* and five minus three is two. But here if you subtract this from the mean you would get a negative number. So five minus six is minus one and five minus eight you get negative three. JO: So above the mean you would have, umm, a total of negative four above the mean. Below the mean you would have six. Ok that is not right. *(pause)* MM: Keep talking	S-CB M-CB M
JO: Now the mean can be added, now add another six in…twenty-four divided by four is six. So yes, you can add the mean back in so that the data set, I mean the mean does not change.	M	JO: Well I know that is not right, it does not make sense. JO: Let me try it this way. *(draws blocks to represent data)* JO: So if the mean is five we have a six and a four and an eight and a, uh, two. Wait, what I was thinking up here. Forget that.	S-CB M
JO: Why does that work. Let me draw a picture. Draw five cubes, seven cubes, and six cubes. In order to make those even we would have to take one from the seven to the six, no, over to the five. All piles will be six. So by adding another pile of six we would not have to move any cubes to make it equal. They will all be six.	S-FS M	JO: Let me draw the line. If we move these blocks around like before, these three here, this one here, we get all stacks equal, five, right. JO: Now let me add the ones below *(subtracts and adds)*…that's four and the one's above *(subtracts and adds)*, negative four. JO: Four plus negative four equal zero. That is what I thought	S-FS

	Pretest	Posttest
Written Artifact		
Rubric Score	— 0 1 2 3	— 0 1 2 3
Comments	• Struggles with center-of-balance • Uses fair share to check problem solution • Relates fair-share and center of balance to sum of direction equal zero	• Incorrectly uses center-of-balance • Uses fair-share and sum of deviations equal zero to solve problem

Participant		Group		Problem	

Pretest Protocol	Code
JO: Let's see if the sum of these four numbers, umm, four of these numbers. Ok. Let's draw five blanks. The mean equals 20. The sum of four of those numbers equals seventy-five. So that means if the mean is twenty some number have to be less than twenty so some will have to be greater than twenty.	S-CB
JO: If the sum of four of those numbers is seventy-five then would the fifth number be less than or greater than twenty. Let's see. Let's take some numbers. Twenty, twenty and twenty is sixty, plus to get to seventy-five you would need a fifteen.	M-FS
JO: So that means, let's see, so if you have three twenties and then a fifteen is five less then twenty, then a fifth number needs to be five more than twenty, needs to be twenty-five. Because, it would all even out, because twenty-five is five more than twenty and fifteen is five less than twenty and you want them to all even out to twenty.	M-CB
JO: Let's see if that works… *(adds numbers and divides)* The value of the fifth number would be twenty-five.	M
Written Artifact: 25 20 20 20 15 = 20 25 20 20 20 15 100 ÷ 5 = 20	
Rubric Score: — 0 1 2 3	
Comments: Both Conceptual/explanations	

Posttest Protocol	Code
JO: Ok, so the sum of three of these numbers is 90. Let's just take thirty and thirty and thirty. JO: The mean of the four numbers is thirty, oh, so the value of the fourth number would be, uh, thirty because they would all be the same.	M-FS
JO: is that right, thirty, sixty, ninety one twenty, divided by four, is thirty. JO: Yea so it the mean, in order for the mean to be thirty they all have to be equal, so the last number would have to be thirty.	M
Written Artifact: 30 30 30 – 30	
Rubric Score: — 0 1 2 3	
Comments:	

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